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Tree-Level N-Point Amplitudes in String Theory

Thesis for the degree of Master of Science

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Abstract

This thesis reviews the method of [1] for computing the full colour ordered N-point open superstring amplitude using the Pure Spinor formalism. We introduce relevant elements of super Yang-Mills theory and examine the basics of the Pure Spinor formalism, with a focus on tools for amplitude computation. We then define a series of objects with increasingly useful BRST transformation properties, which greatly simplify the calculations, and show how these properties can be determined using a diagrammatic method. Finally, we use the explicit four- and five-point amplitude computations as stepping stones to compute the general N-point amplitude, which factors into a set of kinematic integrals multiplying SYM subamplitudes.

Layman's Summary

In particle physics, one of the most common and important questions asked is: "What will happen if a given set of particles interacts?". Typically there is no single answer to this question. Due to the stochastic nature of the quantum world, there are often multiple possible outcomes which may occur with some probability. These probabilities are further dependent on the energies and momenta of the incoming particles. All of this information is encoded in the so-called probability amplitude for that particular interaction. Experimentally, we typically examine the possible outcomes using particle accelerators, but we can also predict them using physical theories and models.

The current standard model of particle physics has been wildly successful in matching the predicted amplitudes to observed experimental results. However, the standard model is incomplete. In particular, it does not seem capable of taking into account the force of gravity. String theory is the most widely accepted solution to this problem of quantum gravity. In string theory, different particles are considered to be different vibrations of more fundamental one-dimensional strings, which may be open lines or closed loops. This description leads us to ask about string interactions, and again we find ourselves concerned with amplitudes.

Amplitudes are typically computed using perturbation theory, where increasingly complex processes contribute with decreasing importance to the overall amplitude. In this thesis, we will concern ourselves with the leading order ("tree level") amplitudes, which contribute most significantly and are also the easiest to calculate. Despite being the simplest level, calculating tree-level amplitudes has nonetheless posed significant difficulties for string theorists interested in increasingly large numbers of incoming and/or outgoing strings, which can be collectively referred to as "points".

String theory can be mathematically phrased in several different ways; these phrasings are referred to as "formalisms". Ideally, different formalisms predict the same physical outcomes, but they lend themselves to simplifying different types of calculations. The recently developed Pure Spinor formalism is particularly effective for calculating probability amplitudes. In this thesis, we will review the development of useful objects and tools within the formalism. Combined with tools and results from other areas of physics, these developments will culminate in our calculation of the probability amplitude for an interaction involving an arbitrary number (N) of open strings.

Populärvetenskaplig Sammanfattning

Inom partikelfysiken är en av de vanligaste och viktigaste frågorna: “Vad händer om en given mängd partiklar interagerar med varandra?”. I allmänhet finns det inte något enkelt svar på denna fråga. På grund av den kvantmekaniska världens stokastiska beteende finns det ofta flera slutresultat som kan inträffa med olika sannolikheter. Dessa sannolikheter är i sin tur beroende på de inkommande partiklarnas energi och rörelsemängd. All denna information finns kodad i den så kallade sannolikhetsamplituden för en viss interaktion. Experimentellt går det vanligtvis att studera de möjliga slutprodukterna genom att använda partikelacceleratorer, men det är också möjlig att förutse dem med hjälp av fysikaliska teorier och modeller.

Den nuvarande standardmodellen för partikelfysik har varit oerhört framgångsrik med att para ihop förutsedda amplituder med experimentella observationer. Dock är standardmodellen för närvarande inte komplett, då den inte tar hänsyn till gravitationen. Den mest accepterade lösningen till problemet med kvantgravitation är strängteori. Inom strängteorin ses partiklar som olika vibrationer av mer fundamentala, endimensionella strängar, som kan vara både öppna linjer eller slutna cirklar. Även inom strängteorin är frågan om interaktioner viktig, och då är amplituder ännu en gång relevanta.

Vanligtvis beräknas amplituder med hjälp av störningsteori, där mer komplexa företeelser har en mindre inverkan på den totala amplituden. Denna uppsats kommer att befatta sig med ledande amplitudtermer (som på engelska kallas “Tree-level”), det vill säga de termer som bidrar mest till den totala amplituden och är enklast att beräkna. Trots deras relativa enkelhet har sådana amplituder visat sig vara besvärliga för strängteoretiker som är intresserade av interaktioner med ett högt antal strängar, där varje interaktion kallas för en “punkt”.

Strängteorin har flera matematiska uppställningar, så kallade formalismer. I princip ska olika formalismer förutse samma fysikaliska fenomen, men vara lämpade för olika typer av beräkningar. Den nyligen utvecklade “Pure Spinor”-formalismen är särskilt välanpassad för beräkningar av sannolikhetsamplituder. I den här uppsatsen undersöks utvecklingen av användbara objekt och verktyg inom “Pure Spinor”- formalismen. Tillsammans med resultat från andra delar av fysiken kommer denna uppsats att kulminera i en beräkning av sannolikhetsamplituden for en interaktion mellan ett godtyckligt antal (N) öppna strängar.

Acknowledgements

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Contents

Notation and Conventions

Here we will collect notation and conventions to be used throughout the thesis. These notations will often be used without introduction, and conversely any new notations, if not introduced explicitly in the text, will appear in this section.

We will denote spacetime vector indices by $m, n, \ldots = 0, \ldots, 9$ and spacetime spinor indices by $\alpha, \beta, \ldots = 1, \ldots, 16$. We will typically reserve $i, j, k, \ldots$ to use as labels, not indices, meaning they imply no transformation properties unless explicitly stated. When symmetrizing or antisymmetrizing N indices, we define the brackets ( ) and [ ] respectively to contain an overall factor of $1/N!$, such that e.g. $A^{[mn]} \equiv (A^m A^n - A^n A^m)/2!$.

To avoid clutter, we will sometimes suppress both spinor and vector indices when they are contracted. Spinor index suppression will be denoted by round brackets, so e.g. $A_\alpha \gamma_m^{\alpha\beta} B_\beta = (A\gamma_m B)$, and $A_\alpha \gamma_m^{\alpha\beta} = (A\gamma_m)^\beta$. Vector index contractions will be denoted as a dot product: $A^m B_m = A \cdot B$. Where unambiguous, we may also use this notation for higher rank tensors, e.g. $C^{mn} D_{mn} = C \cdot D$.

When taking derivatives of superfields, we will use the convention $\partial_m = k_m$ (not ik_m as usual). We will denote the 16×16 Pauli matrices as $\gamma^m_{\alpha\beta}$. Higher order antisymmetrized products of these matrices will be given by $\gamma^{m_1 m_2 \ldots m_k}_{\alpha\beta} \equiv (\gamma^{[m_1}\gamma^{m_2} \cdots \gamma^{m_k]})_{\alpha\beta}$.

When computing operator produce expansions (OPEs), we will denote "equal up to regular terms" by an arrow ($\rightarrow$). For OPE singularities we will use the shorthand $z_{ij} = z_i - z_j$. Where potentially ambiguous, we denote differentiation variables using $\partial/\partial z_i \equiv \partial_i$. If the limit for an OPE is not explicitly stated, we will assume that the object written to the right in the product approaches the object to the left: "$V(z_1)U(z_2) \rightarrow \ldots$" means "$\lim_{z_2 \rightarrow z_1} V(z_1)U(z_2) \rightarrow \ldots$".

Where α' does not explicitly appear, we will use the convention $\alpha' = 1/2$. It can always be restored by dimensional analysis if needed. We will denote sums of momenta by $k^{12\ldots p} \equiv k^1 + k^2 + \cdots + k^p$. We will make use of dimensionless Mandelstam invariants $s_{12\ldots p} \equiv \alpha'(k^{12\ldots p})^2 = \alpha'(k^1 + k_2 + \cdots + k^p)^2$. In particular, for massless particles and using $\alpha' = 1/2$, $s_{ij} = k^i \cdot k^j$.

1 Introduction

In every modern theory of microscopic physics – from quantum mechanics, to quantum field theory, to string theory – the outcomes of experiments and interactions can only be stated probabilistically. The probability for one set of particles or strings coming together, interacting, and producing a new particular set may depend on the incoming masses, energies, and other variables. The functions of these variables which predict the probability of a given interaction are known as amplitudes.

In both QFT and string theory, amplitudes are typically computed using perturbation theory. Here the simplest ("tree-level") processes, where all involved momenta are fixed, contribute the most significantly, and more complex versions of the interaction can offer corrections. In QFT, these higher order corrections contain loops, whereas in string theory (our main focus) the associated worldsheet contains holes and/or handles. Despite being the simplest level, calculating tree-level amplitudes has nonetheless posed significant difficulties for string theorists interested in increasingly large numbers of incoming and/or outgoing strings, i.e. amplitudes at a large number of points.

In order to be consistent theories with no predicted tachyons, string theories are required to be supersymmetric and in spacetime dimension $D = 10$. This means that we have the same number of bosonic and fermionic degrees of freedom in the equations of motion, and that the theory is invariant under a transformation that relates these degrees of freedom. There have been several equivalent descriptions of superstring theories developed over the years. These different formalisms of superstring theory lend themselves to simplifying different calculations. The Pure Spinor formalism, introduced by Berkovits in 2000 [2], has proven itself to be particularly effective for calculating superstring amplitudes. In fact, and rather remarkably, the general amplitude at *any* number of points was computed in 2011 by Mafra, Schlotterer, Stieberger in [1], using this formalism for open strings. The calculation of the N-point amplitude was the culmination of research by Mafra beginning with the 4-point amplitude in the mid 2000's [3, 4]. The 5-point case soon followed [5]. The 6-point case was analysed in conjunction with Schlotterer, Stieberger *et al.* in [6]. The full N-point case was presented in [1] and subjected to in-depth analysis in [7].

In this thesis, we will review this computation from [1] of the general tree-level N-point amplitude. We will assume familiarity with concepts from quantum field theory, supersymmetry, conformal field theory, and string theory, although important results will be stated explicitly as needed. The interested reader may refer to e.g. [8, 9] for an overview of (super)string theory and relevant aspects of conformal field theory, and the basics of

QFT are covered in [10].

We will begin by reviewing the field equations and symmetries of supersymmetric Yang-Mills theory in $D = 10$. Although SYM is not our primary focus, it will turn out to be an invaluable tool for our calculations. In chapter 3 we will introduce the Pure Spinor formalism by first considering pure spinors in their own right, and then exploring how they are used to construct a consistent superstring theory. We will examine the components of the formalism which are most crucial to our computation, namely the operator product expansions (OPEs) between the various worldsheet fields, and the tree-level amplitude prescription. In chapter 4, we will use the pure spinor formalism to define a series of objects which will prove useful in simplifying our eventual amplitude calculation and in making relevant symmetries more obvious. Finally, in chapter 5, we will introduce some integral manipulation techniques and begin our amplitude calculations. Loosely following the historical development of amplitudes in the Pure Spinor formalism, we will use the explicit 4- and 5-point amplitude computations as stepping stones to computing the full N-point tree-level open superstring amplitude.

2 Elements of Supersymmetric Yang-Mills Theory

We will be making frequent use of $\mathcal{N} = 1$, $D = 10$ supersymmetric Yang-Mills theory (SYM). In this chapter we will review some essential elements of SYM, drawing from similar overviews in [1, 3, 11, 12].

SYM fields are fields on a superspace, a manifold which has both bosonic coordinates X and fermionic (Grassman) coordinates θ. Such fields are often referred to as 'superfields'. We will be considering the $D = 10$ case, with 10 bosonic coordinates X^m, $m = 0, \ldots, 9$, and 16 fermionic coordinates θ^α, $\alpha = 1, \ldots, 16$, which match the $D = 10$ Lorentz vector and spinor. The fundamental fields are the gauge field A_m and its superpartner A_α. We use them to define the supersymmetric and covariant derivatives

$$D_\alpha = \partial_\alpha + \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m \tag{2.1a}$$

$$\nabla_\alpha = D_\alpha + \{A_\alpha, \cdot\} \tag{2.1b}$$

$$\nabla_m = \partial_m + [A_m, \cdot]. \tag{2.1c}$$

The gauge superfields transform under a gauge transformation Ω as

$$\delta A_m = \nabla_m\Omega, \qquad \delta A_\alpha = \nabla_\alpha\Omega, \tag{2.2}$$

which leaves invariant the field strengths

$$\mathcal{F}_{\alpha\beta} = \{\nabla_\alpha, \nabla_\beta\} - \gamma^m_{\alpha\beta}\nabla_m \tag{2.3a}$$

$$\mathcal{F}_{\alpha m} = [\nabla_\alpha, \nabla_m] \equiv (\gamma_m W)_\alpha \tag{2.3b}$$

$$\mathcal{F}_{mn} = [\nabla_m, \nabla_n] = \partial_m A_n - \partial_n A_m + [A_m, A_n], \tag{2.3c}$$

where we defined a new superfield W_α. The superfields appear in the Lagrangian

$$\mathcal{L}_{\mathrm{SYM}} = \mathrm{tr}\left(-\frac{1}{4}\mathcal{F}_{mn}\mathcal{F}^{mn} + \frac{i}{2}A_\alpha(\gamma^m)^{\alpha\beta}\nabla_m A_\beta\right), \tag{2.4}$$

where the trace is over the Lie algebra of the gauge group. This Lagrangian is supersymmetric under the SUSY transformations

$$\delta A_m = i(\xi\gamma_m)^\alpha A_\alpha, \qquad \delta A_\alpha = -\frac{i}{2}(\gamma^{mn}\xi)_\alpha\mathcal{F}_{mn} \tag{2.5}$$

for some fermionic parameter ξ^α. The superfields obey the linearized SYM field equations

$$D_{(\alpha}A_{\beta)} = \gamma^m_{\alpha\beta}A_m \tag{2.6a}$$
$$D_\alpha A_m = (\gamma_m W)_\alpha + k_m A_\alpha \tag{2.6b}$$
$$D_\alpha \mathcal{F}_{mn} = 2k_{[m}(\gamma_{n]}W)_\alpha \tag{2.6c}$$
$$D_\alpha W^\beta = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta \mathcal{F}_{mn}, \tag{2.6d}$$

which as explained in [13] are equivalent to

$$\gamma^{\alpha\beta}_{mnpqr}(D_\alpha A_\beta + igA_\alpha A_\beta) = 0 \tag{2.7}$$

(see [3] for an explicit proof). It is pointed out in [5], that the equations of motion (2.6) imply that

$$k^m(\gamma_m W)_\alpha = 0, \tag{2.8}$$

which will be useful for later computations.

We fix the gauge such that $\theta^\alpha A_\alpha = 0$. Then using the normalization of [14], we can expand the superfields in powers of θ as

$$A_\alpha = \frac{1}{2}a_m(\gamma^m\theta)_\alpha - \frac{1}{3}(\xi\gamma_m\theta)(\gamma^m\theta)_\alpha - \frac{1}{32}F_{mn}(\gamma_p\theta)_\alpha(\theta\gamma^{mnp}\theta) + \ldots \tag{2.9a}$$
$$A_m = a_m - (\xi\gamma_m\theta) - \frac{1}{8}(\theta\gamma_m\gamma^{pq}\theta)F_{pq} + \frac{1}{12}(\theta\gamma_m\gamma^{pq}\theta)(\partial_p\xi\gamma_q\theta) + \ldots \tag{2.9b}$$
$$W^\alpha = \xi^\alpha - \frac{1}{4}(\gamma^{mn}\theta)^\alpha F_{mn} + \frac{1}{4}(\gamma^{mn}\theta)^\alpha(\partial_m\xi\gamma_n\theta) + \frac{1}{48}(\gamma^{mn}\theta)^\alpha(\theta\gamma_n\gamma^{pq}\theta)\partial_m F_{mn} + \ldots \tag{2.9c}$$
$$\mathcal{F}_{mn} = F_{mn} - 2(\partial_{[m}\xi\gamma_{n]}\theta) + \frac{1}{4}(\theta\gamma_{[m}\gamma^{pq}\theta)\partial_{n]}F_{pq} + \frac{1}{6}\partial_{[m}(\theta\gamma_{n]}{}^{pq}\theta)(\xi\gamma_q\theta)\partial_p + \ldots, \tag{2.9d}$$

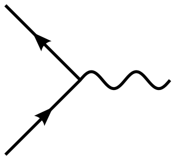
where $a_m(X) = e_m e^{k\cdot X}$, $\xi^\alpha(X) = \chi^\alpha e^{k\cdot X}$ are polarizations and $F_{mn} = 2\partial_{[m}a_{n]}$. Note that we have factors of $e^{k\cdot X}$ and not $e^{ik\cdot X}$ to conform to our convention of $\partial_m = k_m$ when acting on the SYM fields. We recover the other, more common convention simply by sending $k \to ik$.

In SYM, N-point amplitudes $A_{\rm YM}$ can be decomposed into so-called primitive or colour-ordered subamplitudes $\mathcal{A}_{\rm YM}$. At tree level, the decomposition takes the form

$$A_{\rm YM}(1,2,\ldots,N) = g^{N-2}\sum_{\bar\sigma(1,2,\ldots N)} \mathrm{tr}\,(T^{a_1}T^{a_2}\cdots T^{a_N})\,\mathcal{A}_{\rm YM}(1,2,\ldots,N), \tag{2.10}$$

where g is the coupling constant, the T^{a_i} are the generators of the colour group, and $\bar\sigma(1,2,\ldots,N)$ is the set of all non-cyclic permutations of $1,2,\ldots,N$. Note that this is equivalent to summing over *all* permutations of $2,3,\ldots,N$ (i.e. leaving 1 fixed) due to the cyclicity of the trace.

As an aside, the use of YM and not SYM in the labelling of the (sub)amplitudes is not an oversight. Although we are in the supersymmetric case, Lorentz invariance requires fermions to couple to vectors only through vertices of the form of Fig. 2.1. It is easy to convince one's self that a diagram containing only gluonic legs can only contain this vertex as part of a loop, so they do not appear at tree level. This means that for our purposes, results for the YM (sub)amplitudes also hold for the gluonic SYM (sub)amplitudes.



$$\propto \quad \bar{\psi}\not{A}\psi$$

Figure 2.1: The form of the fermion-vector interaction vertex prevents such interactions from appearing at tree-level in gluonic amplitudes.

Now, from the cyclicity of the trace in the decomposition (2.10), we see that the total amplitude is only affected by the part of the subamplitude obeying

$$\mathcal{A}_{\text{YM}}(1,2,\ldots,N) = \mathcal{A}_{\text{YM}}(2,\ldots,N,1), \tag{2.11}$$

and as pointed out in [12] the $\mathcal{A}_{\text{YM}}$ are known to obey this cyclic symmetry property. They also obey other symmetry relations in the indices, in particular the reflection property

$$\mathcal{A}_{\text{YM}}(1,2,\ldots,N) = (-1)^N \mathcal{A}_{\text{YM}}(N,\ldots,2,1) \tag{2.12}$$

and the subcyclic property

$$\sum_{\sigma} \mathcal{A}_{\text{YM}}(1,\sigma(2,3,\ldots,N)) = 0, \tag{2.13}$$

where the sum is over all cyclic permutations $\sigma(2,3,\ldots,N)$ of $2,3,\ldots N$.

3 The Pure Spinor Formalism

The Pure Spinor (PS) formalism was proposed by Berkovits [2] as a method for quantizing the superstring while preserving both manifest super-Poincaré invariance and supersymmetry. In this chapter, we will survey the most important aspects of the PS formalism with respect to our overall goal of computing amplitudes. We will begin by considering pure spinors in their own right, and solving the pure spinor equation. We will then use these objects to write an action, and find the resulting OPEs and relevant composite operators. Finally, we will detail the massless vertex operators and the tree-level amplitude prescription of the formalism.

3.1 Pure Spinors

As the name suggests, the heart of the Pure Spinor formalism is the pure spinor, λ^α, which are bosonic Lorentz spinors obeying the condition

$$\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0. \tag{3.1}$$

where $\gamma^m_{\alpha\beta}$ are the 16×16 symmetric Pauli matrices. We will be concerning ourselves with the case $D = 10$ (so $m = 0, \dots, 9$ and $\alpha, \beta = 1, \dots, 16$). We would like to develop some intuition for the pure spinor by solving this constraint. If we follow [9, 11] and define

$$\gamma_1^\pm = \frac{1}{\sqrt{2}}(\pm\gamma^0 + \gamma^1), \qquad \gamma_a^\pm = \frac{1}{\sqrt{2}}(\gamma^{2(a-1)} \pm i\gamma^{2a-1}), \qquad a = 2, \dots, 5, \tag{3.2}$$

then the pure spinor constraint breaks into $(\lambda\gamma_a^\pm\lambda) = 0,\ a = 1, \dots, 5$. The new matrices obey

$$\{\gamma_a^\pm, \gamma_b^\mp\} = \delta_{ab}, \qquad \{\gamma_a^\pm, \gamma_b^\pm\} = 0. \tag{3.3}$$

Now, from these relations we see that $(\gamma^-)^2 = 0$, meaning we can find a spinor λ^{-----} such that

$$\gamma_a^- \lambda^{-----} = 0 \qquad \forall\ a, \tag{3.4}$$

since the various γ^- all anticommute. Now we extend this notation by defining $\lambda^{+----} = \gamma_1^+\lambda^{-----}$, $\lambda^{+-+--} = \gamma_3^+\gamma_1^+\lambda^{-----}$, etc. Note that we always order the γ with lower indices to the right. The γ_a will annihilate a spinor if the a^{th} label $\pm$ matches its own, and will flip the sign of the label if it does not (potentially with a sign change from anticommutation to preserve the γ matrix ordering). Anti-Weyl spinors are those with an odd number of + labels, whereas an even number (or 0) are Weyl spinors, i.e. multiplication by a γ_a sends Weyl $\to$ anti-Weyl and *vice versa*.

Let us focus on the Weyl spinors. The general Weyl spinor will be some combination of λ with 0, 2, or 4 + labels. We define these basis elements to be

$$\lambda^{-----} \equiv \lambda^-, \qquad \gamma_b^+\gamma_a^+\lambda^- \equiv \lambda_{ab}, \qquad \frac{1}{4!}\varepsilon^{abcde}\gamma_e^+\gamma_d^+\gamma_c^+\gamma_b^+\lambda^- \equiv \lambda^a, \tag{3.5}$$

where λ^- is normalized such that $\lambda^-\lambda^{+++++} = 1$. Note that λ_{ab} is antisymmetric, meaning that the general Weyl spinor has $1 + 10 + 5 = 16$ degrees of freedom, as expected in $D = 10$. These degrees of freedom actually correspond to breaking the (Wick rotated) $SO(10)$ symmetry down into $U(5) = U(1) \times SU(5)$, as pointed out by Berkovits in [2, 13]. The three objects (3.5) transform in the $\mathbf{1_{5/2}}$, $\overline{\mathbf{10}}_{\mathbf{1/2}}$, and $\mathbf{5_{-3/2}}$ representations of $SU(5)_{U(1)}$ respectively.

Now we solve the PS constraints $(\lambda\gamma_a^\pm\lambda) = 0$. Write the general pure spinor as

$$\lambda = u_-\lambda^- + u^{ab}\lambda_{ab} + u_a\lambda^a. \tag{3.6}$$

As pointed out in [11] the only non-zero contractions of our $U(5)$ variables available are

$$\lambda^-\gamma_a^+\lambda^b = \delta_a^b \tag{3.7a}$$

$$\lambda_{ab}\gamma_c^+\lambda_{de} = \varepsilon_{abcde} \tag{3.7b}$$

$$\lambda^a\gamma_b^-\lambda_{cd} = 2\delta^a_{[c}\delta_{d]b}, \tag{3.7c}$$

so the + equation becomes

$$\begin{aligned}(\lambda\gamma_a^+\lambda) &= (u_-\lambda^- + u^{bc}\lambda_{bc} + u_d\lambda^d)\gamma_a^+(u_-\lambda^- + u^{ef}\lambda_{ef} + u_g\lambda^g)\\ &= 2u^-u_b(\lambda^-\gamma_a^+\lambda^b) + u^{bc}u^{de}(\lambda_{bc}\gamma_a^+\lambda_{de})\\ 0 &= 2u^-u_a + \varepsilon_{abcde}u^{bc}u^{de},\end{aligned} \tag{3.8}$$

so we find

$$u_a = \frac{1}{2u^-}\varepsilon_{abcde}u^{bc}u^{de}, \qquad u^- \neq 0. \tag{3.9}$$

Fortunately this also solves the $-$ equation, which reduces to

$$0 = (\lambda\gamma_a^-\lambda) = (u_-\lambda^- + u^{bc}\lambda_{bc} + u_d\lambda^d)\gamma_a^-(u_-\lambda^- + u^{ef}\lambda_{ef} + u_g\lambda^g) = 2u_au^{ab}. \tag{3.10}$$

So we find that the pure spinor is parametrized by a number u^- and an antisymmetric matrix u^{ab} for a total of $1 + 10 = 11$ independent degrees of freedom. This result can also be arrived at using a creation-operator description as in [3]. Note that in the literature (e.g. [3, 13]) sometimes $u^- \neq 0$ is parametrized as $u^- = e^s$ for some number s.

3.2 PS Action and OPEs

The full action for the type IIB superstring in the PS formalism, using the normalization of [1], is

$$S_{\text{full}} = \frac{1}{2\pi}\int \mathrm{d}^2z\left(\frac{1}{2}\partial X^m\bar{\partial}X_m + p_\alpha\bar{\partial}\theta^\alpha - \omega_\alpha\bar{\partial}\lambda^\alpha + \bar{p}_\alpha\partial\theta^\alpha - \bar{\omega}_\alpha\partial\lambda^\alpha\right). \tag{3.11}$$

Here we have bosonic worldsheet fields X^m, λ^α, ω_α, and $\bar{\omega}_\alpha$, and fermionic fields θ^α, p_α, and $\bar{p}_\alpha$.

Since it is our goal to consider only open string amplitudes we can neglect the right-movers and just focus on the left-moving portion

$$S = \frac{1}{2\pi}\int \mathrm{d}^2 z \left(\frac{1}{2}\partial X^m \bar{\partial} X_m + p_\alpha \bar{\partial}\theta^\alpha - \omega_\alpha \bar{\partial}\lambda^\alpha\right), \tag{3.12}$$

which is supersymmetric under

$$\begin{gathered} \delta X^m = \frac{1}{2}(\varepsilon\gamma^m\theta), \qquad \delta\theta^\alpha = \varepsilon^\alpha, \qquad \delta p_\alpha = -\frac{1}{2}(\varepsilon\gamma^m)_\alpha \partial X_m + \frac{1}{8}(\varepsilon\gamma_m\theta)(\partial\theta\gamma^m)_\alpha, \\ \delta\lambda^\alpha = \delta\omega_\alpha = 0, \end{gathered} \tag{3.13}$$

for a small fermionic parameter ε^α and bosonic transformation δ. It is also gauge invariant under

$$\delta\omega_\alpha = \Lambda_m(\gamma^m\lambda)_\alpha \tag{3.14}$$

for arbitrary Λ_m thanks to the PS condition (3.1). Crucial to the development of the formalism is the BRST operator

$$Q = \oint \lambda^\alpha d_\alpha, \tag{3.15}$$

where

$$d_\alpha = p_\alpha - \frac{1}{2}(\gamma^m\theta)_\alpha \partial X_m - \frac{1}{8}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta) \tag{3.16}$$

is the Green-Schwartz (GS) constraint, which in this case is not constrained to vanish. This action and BRST operator began in [2] as an educated guess, but Berkovits has since presented a derivation of both the PS and the GS actions and BRST operators as different gauges of a purely bosonic, classical action starting from superparticle considerations in [15]. We also have the supersymmetric momentum combination

$$\Pi^m = \partial X^m + \frac{1}{2}(\theta\gamma^m\partial\theta). \tag{3.17}$$

The action (3.12) yields the standard free boson and bc OPEs:

$$X^m(z_1)X^n(z_2) \to -\frac{\alpha'}{2}\eta^{mn}\ln z_{12}, \qquad p_\alpha(z_1)\theta^\beta(z_2) \to \frac{\delta_\alpha^\beta}{z_{12}}. \tag{3.18}$$

Using Wick's theorem we can then compute more OPEs:

$$\begin{gathered} \Pi^m(z_1)\Pi^n(z_2) \to -\frac{\eta^{mn}}{(z_{12})^2}, \qquad \Pi^m(z_1)X^n(z_2) \to -\frac{\eta^{mn}}{z_{12}} \\ d_\alpha(z_1)d_\beta(z_2) \to -\frac{\gamma^m_{\alpha\beta}\Pi^m}{z_{12}}, \qquad d_\alpha(z_1)\theta^\beta(z_2) \to \frac{\delta_\alpha^\beta}{z_{12}}, \qquad d_\alpha(z_1)\Pi^m(z_2) \to \frac{(\gamma^m\theta)_\alpha}{z_{12}}. \end{gathered} \tag{3.19}$$

There is significant subtlety which comes into play regarding the ghost number current

$$J = \omega_\alpha\lambda^\alpha \tag{3.20}$$

and the Lorentz current, which we write as

$$M^{mn} = \Sigma^{mn} + N^{mn}, \qquad N^{mn} = \frac{1}{2}(\lambda\gamma^{mn}\omega), \tag{3.21}$$

where Σ^{mn} is the normal Lorentz current for the matter, and N^{mn} is that of the ghosts. As detailed in [3, 16], the key issue is that the $\omega\lambda$ CFT is constrained by the PS condition, which makes the OPEs resulting from it non-trivial to calculate. The calculations can be performed by breaking the symmetry down to the $U(5)$ version of section 3.1, including defining new fields ω_-, ω^{ab}, and ω_a in the same way as the broken down λ. The idea is to use the solution to the pure spinor constraint, and choose the gauge $\omega^a = 0$ using (3.14), to write a new action for the ghosts in terms of fields composed of λ^+ and u^{ab}. This allows us to compute a fairly large number of $U(5)$ OPEs, which are listed in [3]. Luckily, these can be reassembled into the desired Lorentz covariant form. This process is laid out in great detail in [3, 16]. In the end, we get the OPEs

$$N^{mn}(z_1)\lambda^\alpha(z_2) \to -\frac{1}{2}\frac{(\lambda\gamma^{mn})^\alpha}{z_{12}}, \qquad N^{mn}(z_1)N_{pq}(z_2) \to \frac{4}{z_{12}}N^{[m}{}_{[p}\delta^{n]}_{q]} - \frac{6}{(z_{12})^2}\delta^n_{[p}\delta^n_{q]} \tag{3.22}$$

and

$$J(z_1)\lambda^\alpha(z_2) = \frac{\lambda^\alpha}{(z_{12})} \tag{3.23}$$

such that λ has ghost number 1. We also find the stress-energy tensor

$$T(z) = -\frac{1}{2}\Pi^m\Pi_m - d_\alpha\partial\theta^\alpha + \omega_\alpha\partial\lambda^\alpha. \tag{3.24}$$

The central charge for this stress-energy tensor receives a contribution of $+10$ from the X^m degrees of freedom and -32 from the fermions. We have already established that the pure spinors have 11 degrees of freedom; they contribute $+22$ to the central charge for a grand total of zero. So the theory is anomaly free as required. Finally, we also get the action of d and Π on general superfields $f(X,\theta)$ containing only zero modes of θ:

$$d_\alpha(z_1)f(z_2) \to \frac{D_\alpha f}{z_{12}}, \qquad \Pi^m(z_1)f(z_2) \to -\frac{k^m f}{z_{12}}. \tag{3.25}$$

In particular, these are the two OPEs obeyed by the SYM fields.

3.3 Vertex Operators and Amplitudes

The massless vertex operators were introduced with the PS formalism in [2]. They are

$$V(z) = \lambda^\alpha A_\alpha \tag{3.26}$$

in unintegrated form and

$$U(z) = \partial\theta^\alpha A_\alpha + A_m\Pi^m + d_\alpha W^\alpha + \frac{1}{2}N_{mn}\mathcal{F}^{mn} \tag{3.27}$$

in integrated form. These vertex operators have ghost numbers 1 and 0, respectively, just like in bosonic string theory[1]. Here $A_\alpha = A_\alpha(X,\theta)$, $A_m = A_m(X,\theta)$, $W^\alpha = W^\alpha(X,\theta)$, and $\mathcal{F}^{nm} = \mathcal{F}^{nm}(X,\theta)$ are a set of suggestively-named superfields. We would like to constrain these superfields such that we have the usual relations

$$QV = 0, \qquad QU = \partial V. \tag{3.28}$$

Using our OPEs from section 3.2, we act with Q on U. The first term yields

$$\begin{aligned}
Q(\partial\theta^\beta(z_2))A_\beta(z_2) = &\oint \mathrm{d}z_1\, \frac{1}{2\pi i}\lambda^\alpha(z_1)d_\alpha(z_1)\partial_2\theta^\beta(z_2)A_\beta(z_2) \\
\to &\oint \mathrm{d}z_1\, \frac{1}{2\pi i}\lambda^\alpha(z_1)\left(\partial_2\left(\frac{\delta^\beta_\alpha}{z_{12}}\right)A_\beta(z_2) - \partial_2\theta^\beta(z_2)\frac{D_\alpha A_\beta(z_2)}{z_{12}}\right) \\
= &\oint \mathrm{d}z_1\, \frac{1}{2\pi i}(\lambda^\alpha(z_2) + \partial_2\lambda^\alpha(z_2)z_{12} + \dots) \\
&\qquad \times \left(\frac{A_\alpha(z_2)}{z_{12}^2} - \partial_2\theta^\beta(z_2)\frac{D_\alpha A_\beta(z_2)}{z_{12}}\right) \\
= &\,(\partial\lambda^\alpha)A_\alpha - \lambda^\alpha(\partial\theta^\beta)D_\alpha A_\beta.
\end{aligned} \tag{3.29}$$

Similarly, the following terms yield

$$Q(A_m\Pi^m) = \lambda^\alpha\Pi^m D_\alpha A_m + (\lambda\gamma^m\partial\theta)A_m \tag{3.30}$$

$$Q(d_\beta W^\beta) = -(\lambda\gamma_m W)\Pi^m - \lambda^\alpha d_\beta D_\alpha W^\beta \tag{3.31}$$

$$\frac{1}{2}Q(N_{mn}\mathcal{F}^{mn}) = \frac{1}{2}N_{mn}\lambda^\alpha D_\alpha\mathcal{F}^{mn} + \frac{1}{4}(\lambda\gamma_{mn}d)F^{mn}. \tag{3.32}$$

Combining these terms, we get

$$\begin{aligned}
QU = &(\partial\lambda^\alpha)A_\alpha - \lambda^\alpha(\partial\theta^\beta)(D_\alpha A_\beta - \gamma^m_{\alpha\beta}A_m) + \lambda^\alpha\Pi^m(D_\alpha A_m - (\gamma_m W)_\alpha) \\
&- \lambda^\alpha d_\beta(D_\alpha W^\beta - \frac{1}{4}(\gamma_{mn})_\alpha{}^\beta\mathcal{F}^{mn}) + \frac{1}{2}N_{mn}\lambda^\alpha(D_\alpha\mathcal{F}^{mn}).
\end{aligned} \tag{3.33}$$

This is equal to

$$\partial V = \partial(\lambda^\alpha A_\alpha) = (\partial\lambda^\alpha)A_\alpha + \lambda^\alpha(\partial\theta^\beta)\partial_\beta A_\alpha + \lambda^\alpha(\partial X^m)\partial_m A_\alpha \tag{3.34}$$

only if the superfields A, $\mathcal{F}$, and W obey precisely the SYM field equations (2.6)! This setup means that SYM appears very naturally in the PS formalism, allowing for the use of results from this well-studied theory. The PS formalism has also been fruitfully applied to derive SYM results as introduced in [17] and reviewed in e.g. [11]. Note that the N_{mn} vanishes after writing N in terms of λ and using the the PS relation from [2]:

$$(\gamma^m\lambda)_\alpha(\gamma_m\lambda)_\beta = -\frac{1}{2}\gamma^m_{\alpha\beta}(\lambda\gamma_m\lambda) = 0. \tag{3.35}$$

[1] The vertex operators of bosonic string theory are explained in detail in section 2.8 of [8]

Similarly, $QV = 0$ if A_α obeys the linearized SYM equations (2.7). So demanding the usual vertex operator relations is equivalent to putting the coefficient superfields on the SYM mass shell. In all further calculations we will thus assume that any SYM fields appear on-shell, unless explicitly stated otherwise. As an aside, it is worth noting for ease of OPE computations that we can rewrite ∂V as

$$\partial V = (\partial\lambda^\alpha)A_\alpha + \Pi^m k_m V + \partial\theta^\alpha D_\alpha V. \tag{3.36}$$

In analogy to bosonic string theory, colour ordered tree-level amplitudes at N points are given by

$$\begin{aligned}&\mathcal{A}(1,2,\ldots,N)\\ &\quad= \left\langle V^1(0)V^{N-1}(1)V^N(\infty)\int \mathrm{d}z_2 U^2(z_2)\int \mathrm{d}z_3 U^3(z_3)\cdots\int \mathrm{d}z_{N-2}U^{N-2}(z_{N-2})\right\rangle,\end{aligned} \tag{3.37}$$

where we integrate over the region $z_1 = 0 \leq z_2 \leq \cdots \leq z_{N-2} \leq z_{N-1} = 1$. Here we have used the $SL(2,\mathbb{R})$ invariance of the open worldsheet to fix $(z_1, z_{N-1}, z_N) = (0, 1, \infty)$. The correlator brackets $\langle\ \rangle$ are defined to be non-zero only for, and normalized by, the combination of zero modes

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1. \tag{3.38}$$

This combination is easily seen to be BRST closed using our OPEs and the pure spinor condition, so BRST exact terms decouple from physical states (see e.g. [16] for an in-depth discussion of the decoupling). Furthermore, it was proven in [2] that despite the explicit θ dependence, this measure is both supersymmetric and gauge invariant. That this is the correct object to absorb the zero modes was an *ansatz* made in analogy to bosonic string theory, where the zero mode prescription is given by $\langle c\partial c\partial^2 c\rangle = 1$, i.e. we normalize using a vertex operator at ghost number $+3$. The authors of [1] point out that the combination (3.38) is the unique element in the cohomology of Q at ghost number 3.

Note that because BRST exact terms decouple and Q acts like a differential operator, we can use the correlator to perform "BRST integration by parts". In particular, consider a fermionic object T and a bosonic object D. Since Q is fermionic, its action on T and D is

$$QD = [Q, D], \qquad QT = \{Q, T\}\,. \tag{3.39}$$

Then, acting on the products DT and TD gives

$$\begin{aligned}Q(TD) &= \{Q, TD\} = \{Q, T\}\,D - T[Q, D] = (QT)D - T(QD),\\ Q(DT) &= \{Q, DT\} = [Q, D]T + D\,\{Q, T\} = (QD)T + D(QT).\end{aligned} \tag{3.40}$$

So, when we BRST integrate by parts, we get

$$\begin{aligned}\langle Q(TD)\rangle = 0 &\implies \langle(QT)D\rangle = \langle T(QD)\rangle,\\ \langle Q(DT)\rangle = 0 &\implies \langle(QD)T\rangle = -\langle D(QT)\rangle.\end{aligned} \tag{3.41}$$

4 Building Blocks: Laying a Foundation

In this chapter we follow [1] in defining a series of objects which will aid in the eventual computation of the N-point amplitude. We will make extensive use of several layers of recursively defined superfields, with the aim of elucidating their BRST transformation properties and symmetries. Their explicit computations will rely heavily on the OPEs that we calculated in section 3.2. We will begin with a convenient definition of a set of recursively defined, composite superfields L. We will then go through significant effort to remove BRST trivial parts of L while still maintaining essential properties, leaving us with the so-called BRST building blocks T. Using combinations of these, we will finally construct objects known as the supersymmetric Berends-Giele currents M, which are the actual objects appearing in amplitudes. As a point of interest we also consider an expression of super Yang-Mills subamplitudes in terms of the BG currents.

4.1 Recursive Superfields from Vertex Operators

We begin by defining the superfields L, up to BRST exact terms, by

$$\lim_{z_2 \to z_1} V^1(z_1)U^2(z_2) \to \frac{L_{21}}{z_{21}}, \qquad \lim_{z_p \to z_1} L_{2131\ldots(p-1)1}(z_1)U^p(z_p) \to \frac{L_{2131\ldots p1}}{z_{p1}}. \tag{4.1}$$

We can generalize this notation to include the single poles resulting from taking the limits in other orders, similar to [5, 6]. To do so we use the labels of the $L_{ijk\ell mn\ldots}$ to mean that we take the limits $z_i \to z_j$, then $z_k \to z_\ell$, then $z_m \to z_n$, etc., always involving one V and a series of U:

$$\cdots \lim_{z_f \to z_g} \lim_{z_c \to z_d} \lim_{z_a \to z_b} V^i(z_i)U^j(z_j)U^k(z_k)U^\ell(z_\ell)\cdots \to \frac{L_{abcdfg\ldots}}{z_{ab}z_{dc}z_{fg}\cdots}. \tag{4.2}$$

Here the labels $abc\ldots$ must be such that after all the limits are taken, all of the vertex operators end up converging on the same point. For example, we could have $a = i,\ b = f = j,\ c = k$, and $d = g = \ell$. Then the z limits are: first $i \to j$, then $k \to \ell$, and finally $j \to \ell$. After taking these limits in order we are left with $L_{ijk\ell j\ell}(z_\ell)/z_{ij}z_{k\ell}z_{j\ell}$. We will leave further discussion of these generalized L to chapter 5, instead focusing for now on the simpler $L_{2131\ldots}$.

Now, we would like to explicitly compute L_{21}. Using our OPEs and the SYM equations of motion (2.6), we get

$$\begin{aligned}
z_{21}V^1(z_1)U^2(z_2) &= z_{21}(\lambda^\alpha A^1_\alpha)(\partial\theta^\beta A^2_\beta + \Pi^m A^2_m + d_\beta W^{2\beta} + \frac{1}{2}\mathcal{F}^2_{mn}N^{mn}) \\
&\to 0 - \lambda^\alpha k^{1m} A^1_\alpha A^2_m - \lambda^\alpha (D_\beta A^1_\alpha) W^{2\beta} - \frac{1}{4} A^1_\beta \lambda^\alpha \lambda^{mn}{}_\alpha{}^\beta \mathcal{F}^2_{mn} \\
&= -V^1(k^1\cdot A^2) - \lambda^\alpha(\gamma^m_{\alpha\beta}A^1_m - D_\alpha A^1_\beta)W^{2\beta} - \lambda^\alpha A^1_\beta D_\alpha W^{2\beta} \\
&= -V^1(k^1\cdot A^2) - A^1_m(\lambda\gamma^m W^2) + \lambda^\alpha D_\alpha(A^1_\beta W^{2\beta}) \\
&= -V^1(k^1 A^2) - A^1_m(\lambda\gamma^m W^2) + Q(A^1 W^2).
\end{aligned} \tag{4.3}$$

Note the presence of the BRST exact term $Q(A^1\cdot W^2)$. Since we are taking L only up to BRST exact terms, we drop this term and simply write

$$L_{21} = -V^1(k^1\cdot A^2) - A^1_m(\lambda\gamma^m W^2). \tag{4.4}$$

Failure to remove this explicit term BRST-exact terms will yield a cascade of extraneous terms as the definition is recursively applied, breaking the transformation properties we are about to compute.

We are primarily interested in the BRST properties of these superfields, and dropping this term does not affect the cohomology. Note that this reduced expression with no BRST exact terms is the one that is used in the recursive definition of the higher fields. Explicit expressions for higher orders in L and many of the following recursive superfields are given throughout [1]. For later use, we also include the next rank of L explicitly:

$$\begin{aligned}
L_{2131} &= z_{31}\lim_{z_3\to z_1} L_{21}(z_1)U^3(z_3) \\
&= -L_{21}(k^{12}\cdot A^3) - \big(L_{31} + V^1(k^1\cdot A^3)\big)(k^1\cdot A^2) \\
&\quad + \big(L_{32} + V^2(k^2\cdot A^3)\big)(k^2\cdot A^1) - (\lambda\gamma^m W^3)\big((W^1\gamma_m W^2) - k^2_m(A^1\cdot A^2)\big).
\end{aligned} \tag{4.5}$$

Now, how does L_{21} behave under the action of Q?

$$\begin{aligned}
QL_{21} &= \oint \lambda^\alpha d_\alpha(-V^1(k^1\cdot A^2) - A^1_m(\lambda\gamma^m W^2)) \\
&\to V^1 k^1_m \lambda^\alpha(D_\alpha A^{2m}) - \lambda^\alpha(D_\alpha A^1_m)(\lambda\gamma^m W^2) - A^1_m\lambda^\alpha\lambda^\beta D_\alpha(\gamma^m W^2)_\beta.
\end{aligned} \tag{4.6}$$

Starting from the last term, we get

$$-A^1_m\lambda^\alpha\lambda^\beta D_\alpha(\gamma^m W^2)_\beta = -A^1_m\lambda^\alpha\lambda^\beta D_\alpha(D_\beta A^{2m} + k^{2m}A^2_\beta) = -(A^1\cdot k^2)\lambda^\alpha\lambda^\beta\gamma^n_{\alpha\beta}A^2_n = 0, \tag{4.7}$$

where we used the SYM equations, that $D_\alpha D_\beta$ is antisymmetric, and the PS condition. Similarly, with the help of (3.35), we see that

$$\begin{aligned}
-\lambda^\alpha(D_\alpha A^1_m)(\lambda\gamma^m W^2) &= -\lambda^\alpha\lambda^\beta((\gamma^m W^1)_\alpha + k^{1m}A^1_\alpha)(\gamma_m W^2) \\
&= -\lambda^\alpha A^1_\alpha k^{1m}\lambda^\beta(\gamma_m W^2)_\beta \\
&= -V^1 k^{1m}\lambda^\beta(D_\beta A^2_m - k^2_m A^2_\beta) \\
&= -V^1 k^1_m \lambda^\alpha(D_\alpha A^{2m}) + V^1V^2(k^2\cdot k^2).
\end{aligned} \tag{4.8}$$

So the first term of (4.7) is cancelled. Since we are dealing with massless fields, $k^1 \cdot k^2 = \frac{1}{2}(k^1+k^2)^2 = s_{12}$, and we are left with

$$QL_{21} = s_{12} V^1 V^2. \tag{4.9}$$

This forms the base case for the general recursive action of Q, which is

$$QL_{2131\ldots p1} = \lim_{z_p \to z_1} z_{p1} \left((QL_{2131\ldots(p-1)1})(z_1) U^p(z_p) - L_{2131\ldots(p-1)1}(z_1) \partial V^p(z_p) \right) \tag{4.10}$$

(no sum over p), as is easily seen from recalling that $QU = \partial V$ and noting that both Q and L are fermionic. To evaluate the second term in this expression, we use our OPEs and (3.36) to compute that

$$V^i(z_i) \partial V^p(z_p) = (\lambda^\alpha A^i_\alpha) \left((\partial \lambda^\beta) A^p_\beta + \Pi^\ell k^p_\ell V^p + \partial \theta^\beta D_\beta V^p \right) \to -\frac{s_{pi}}{z_{pi}} (V^i V^p)(z_i) \tag{4.11}$$

(no sum over i, p), and

$$\begin{aligned} U^j(z_i) \partial V^p(z_p) = {} & (\partial \theta^\alpha A^j_\alpha + A^j_m \Pi^m + d_\alpha W^{j\alpha} + \frac{1}{2} N_{mn} \mathcal{F}^{j\,mn}) \\ & \times \left((\partial \lambda^\beta) A^p_\beta + \Pi^\ell k^p_\ell V^p + \partial \theta^\beta D_\beta V^p \right) \\ & \to -\frac{s_{pj}}{z_{pi}} (U^j V^p)(z_i). \end{aligned} \tag{4.12}$$

So

$$\begin{aligned} \lim_{z_3 \to z_1} -z_{31} L_{21}(z_1) \partial V^3(z_3) = {} & - z_{31} (V^1 U^2) \partial V^3 \\ & \to -z_{31} \left(-\frac{s_{31}}{z_{31}} V^1 V^3\, U^2 - V^1 \frac{s_{32}}{z_{31}} U^2 V^3 \right) \\ & = (s_{13} + s_{23}) L_{21} V^3. \end{aligned} \tag{4.13}$$

Finally, using the definition of the higher order superfields (4.1) one can inductively show

$$-\lim_{z_p \to z_1} z_{p1} L_{2131\ldots(p-1)1}(z_1) \partial V^p(z_p) = \sum_{j=1}^{p-1} s_{jp} L_{2131\ldots(p-1)1} V^p \tag{4.14}$$

(no sum over p). Already we see kinematic factors appearing from our treatment.

In calculations to follow, our recursive fields do not exist in isolation, but in various combinations, which may have interesting BRST behaviour. For example, since V^1 and V^2 anticommute,

$$Q(L_{21} + L_{12}) = s_{12}(V^1 V^2 + V^2 V^1) = 0, \tag{4.15}$$

so this combination is BRST closed. It turns out that it is also exact:

$$L_{21} + L_{12} = -\lambda^\alpha ((\gamma_m W^1)_\alpha + k^1_m A^1_\alpha) A^{2m} + (1 \leftrightarrow 2) = -\lambda^\alpha D_\alpha (A^1_m A^{2m}) = -Q(A^1 \cdot A^2). \tag{4.16}$$

Since this is a factor that will show up frequently, we define for convenience

$$D_{ij} = D_{ji} = A^i \cdot A^j. \tag{4.17}$$

We can also view this as the L having an explicitly exact part by writing $L_{21} = L_{[21]} + L_{(21)}$, with $L_{(21)}$ exact.

In fact, all BRST closed combinations of $L_{ijik\dots}$ are exact. The authors of [1] point out that using the stress-energy tensor (3.24), the conformal weight of any $L_{ijik\cdots in}$ is $(k^i + k^j + k^k + \cdots + k^n)^2$. This is non-zero for massless strings N points as long as $n < N - 1$.

Now, we know from bosonic string theory that for a b ghost,

$$\{Q, b\} = T, \qquad \{Q, b_0\} = L_0 \tag{4.18}$$

where L_0 and b_0 are the zero modes of the stress-energy tensor and the b ghost (see chapter 4 of [8] for a review of this point). Primary operators (including our $L_{ijik\dots}$) are eigenstates of L_0 with eigenvalue h, the conformal weight of the operator. Now suppose that we have an operator ψ which is BRST closed. Then

$$h\psi = L_0\psi = \{Q, b_0\}\,\psi. \tag{4.19}$$

But since ψ is closed, $\{Q, b_0\}\,\psi = Qb_0\psi$. If $h \neq 0$ then we can invert the equation:

$$\psi = Q\left(\frac{1}{h}b_0\psi\right). \tag{4.20}$$

So if $h \neq 0$, then closedness of ψ implies its exactness.

The Pure Spinor formalism does not have a b ghost as a fundamental field, so it is not immediately obvious that this result applies here. However, it is possible to construct a composite object obeying the same relations as the b ghost; this was first done by Berkovits to aid in the calculation of multiloop Pure Spinor amplitudes in [18]. The existence of such a field, composite or not, is sufficient to apply this result to our case, and so all closed combinations of $L_{ijik\dots}$ must also be exact.

We would like to spare ourselves the trouble of dealing with this and other exact combinations which will anyway not affect physical results, so we now seek a systematic way to eliminate this BRST exactness from our expressions.

4.2 BRST Building Blocks

It was not necessarily obvious from the first few lines of (4.3) that there would be an explicit BRST-exact part of L_{21} to be removed, and this obscurity only worsens as more recursive steps are added. As such, a procedure is designed in [1] to eliminate these parts systematically and leave us with the so-called BRST-building blocks, $T_{123\dots p}$, which allow us to examine potential BRST closed and exact combinations more systematically.

The idea is to construct the building blocks $T_{123\ldots p}$ such any exact combination of them is explicitly zero, while still maintaining the BRST property (4.10) under the replacement $L_{2131\ldots p1} \to T_{123\ldots p}$. This definition leads to symmetries in the labels of the building blocks, which we will refer to as BRST symmetries.

We have already discovered how to construct the base case, by removing the BRST trivial part (4.16) of L_{21}. Since $L_{(21)}$ is exact and $L_{21} = L_{[21]} + L_{(21)}$, we define

$$T_{12} \equiv L_{21} + \frac{1}{2} QD_{21} = -V^1(k^1 \cdot A^2) - A^1_m(\lambda\gamma^m W^2) + \frac{1}{2} Q(A^1 \cdot A^2). \tag{4.21}$$

From this definition we see that our first building block has the BRST symmetry

$$T_{12} + T_{21} = 0. \tag{4.22}$$

Now we seek to define higher rank building blocks, which we do by means of an intermediary step. We first define $\tilde{T}_{12\ldots p}$ such that $Q\tilde{T}_{12\ldots p}$ contains only $T_{12\ldots(p-1)}$ instead of $L_{2131\ldots(p-1)1}$. Then we define $T_{12\ldots p}$ by removing any BRST exact parts of $\tilde{T}_{12\ldots p}$. To keep our definitions consistent we note that $\tilde{T}_{12} = T_{12}$.

We illustrate this process using the next rank. It is convenient to rewrite L_{2131} (4.5) to the form

$$L_{2131} = -L_{21}(k^{12} \cdot A^3) + (\lambda\gamma^m W^3)(A^1_m(k^1 \cdot A^2) + A^{1n}\mathcal{F}^2_{mn} - (W^1\gamma_m W^2)). \tag{4.23}$$

Then, using (4.10), this has the BRST transformation

$$QL_{2131} = s_{12}(L_{31}V^2 - L_{32}V^1) + (s_{13} + s_{23})L_{21}V^3. \tag{4.24}$$

We define $\tilde{T}$ to obey this relation under the replacements $L \to \tilde{T}$ on the LHS and $L \to T$ on the RHS:

$$Q\tilde{T}_{123} \equiv s_{12}(T_{13}V^2 - T_{23}V^1) + (s_{13} + s_{23})T_{12}V^3. \tag{4.25}$$

From (4.21) and (4.24) we see that

$$\tilde{T}_{123} = L_{2131} + \frac{1}{2}s_{12}(D_{13}V^2 - D_{23}V^1) + \frac{1}{2}(s_{13} + s_{23})D_{12}V^3. \tag{4.26}$$

Now all that remains is to find T_{123} by removing any BRST exact parts of $\tilde{T}_{123}$. It will turn out that $\tilde{T}_{123}$ has two such parts:

$$\tilde{T}_{123} + \tilde{T}_{213} = Q[D_{12}(k^{12} \cdot A^3)] \equiv QR^{(1)}_{123} \tag{4.27}$$

$$\tilde{T}_{123} + \tilde{T}_{231} + \tilde{T}_{312} = Q[D_{12}(k^2 \cdot A^3) + \text{cyclic}(123)] \equiv QR^{(2)}_{123}. \tag{4.28}$$

Removing these parts gives us T_{123}, and the BRST symmetries that go with it:

$$T_{123} = \tilde{T}_{123} - \frac{1}{2}QR^{(1)}_{123} - \frac{1}{3}QR^{(2)}_{[12]3} \equiv \tilde{T}_{123} - QS_{123} \tag{4.29}$$

$$T_{123} + T_{213} = 0 = T_{123} + T_{231} + T_{312}, \tag{4.30}$$

where we defined $S_{123} = \frac{1}{2}R^{(1)}_{123} + \frac{1}{3}R^{(2)}_{123}$ to be the total piece to be removed. We observe that our new rank-3 building block still has the old rank-2 BRST symmetry (4.22) in the first two labels, and that we have gained an entirely new symmetry which uses all three labels. This behaviour, as well as the method to identify the new BRST parts of higher rank $\tilde{T}$, can be explained by a diagrammatic interpretation of the building blocks. In summary, we have succeeded (contingent on identifying the $R^{(\ell)}_{12\ldots p}$) in defining superfields $T_{12\ldots p}$ which have BRST symmetries in their labels, and which obey

$$QT_{12\ldots p} = T_{12\ldots(p-1)}V^p \sum_{j=1}^{p-1} s_{jp} + \lim_{z_1\to z_p} z_{p1} QT_{12\ldots(p-1)}(z_1)U^p(z_p). \tag{4.31}$$

A very useful method from [1] to identify all the BRST-exact pieces of $R^{(\ell)}_{12\ldots p}$ to remove from $\tilde{T}$ (or, equivalently, of identifying the BRST symmetries of the T) is a diagrammatic representation of the T. The key to identifying the building blocks as diagrams lies in the Mandelstam content of their BRST variations. We have seen that

$$QT_{12} = Q\tilde{T}_{12} = QL_{12} = s_{12}V^1V^2 \tag{4.32}$$

contains a factor of s_{12}. Furthermore, we can write (4.25) as

$$QT_{123} = Q\tilde{T}_{123} = s_{123}T_{12}V^3 - s_{12}(T_{23}V^1 - T_{13}V^2 + T_{12}V^3), \tag{4.33}$$

containing both s_{12} and $s_{123} = \frac{1}{2}(k^1+k^2+k^3)^2$. Looking at the general expression (4.31), it is easy to convince one's self that each higher rank $QT_{12\ldots p}$ will contain all the lower rank Mandelstam invariants from $QT_{12\ldots(p-1)}$, and can be written to include the newest rank as well. We can even rewrite the general case to make this explicit:

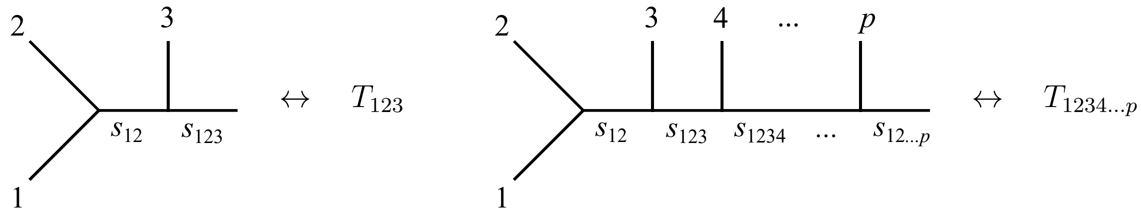
$$QT_{12\ldots p} = s_{12\ldots p}T_{12\ldots(p-1)}V^p - T_{12\ldots(p-1)}V^p \sum_{k=2}^{p-1}\sum_{j=1}^{k-1} s_{jk} + \lim_{z_1\to z_p} z_{p1} QT_{12\ldots(p-1)}(z_1)U^p(z_p). \tag{4.34}$$

This situation is displayed schematically in Fig. 4.1.

$$
\begin{array}{l}
\left.\begin{array}{l}
\left.\begin{array}{l}
\left.\begin{array}{rl} QT_{12}: & s_{12} \\ QT_{123}: & s_{13}+s_{23} \end{array}\right\} s_{123} \\
QT_{1234}: \quad s_{14}+s_{24}+s_{34}
\end{array}\right\} s_{1234} \\
QT_{12345}: \quad s_{15}+s_{25}+s_{35}+s_{45}
\end{array}\right\} s_{12345} \quad \ddots \\
\qquad\qquad \vdots
\end{array}
$$

Figure 4.1: The BRST variation of higher order building blocks contains higher order Mandelstam invariants and all lower orders.

At this point we can uniquely identify each T with a series of labels and the Mandelstam invariants present in its BRST variation. These are exactly the features of a cubic



(a) Our known example (b) The general case

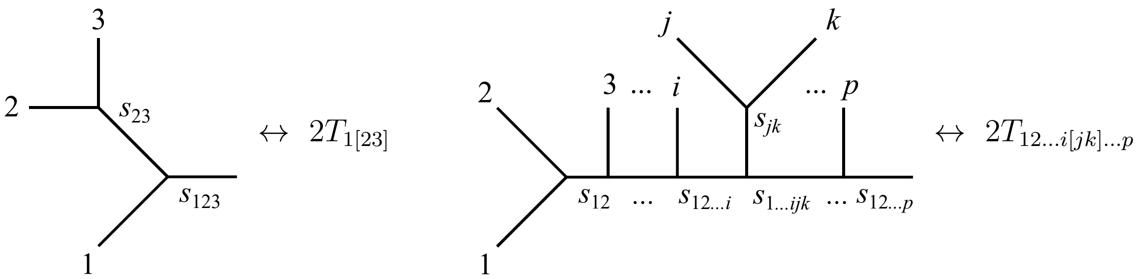
Figure 4.2: BRST building blocks can be associated with cubic diagrams which have the same indices and Mandelstam invariants.

Feynman-type diagram! If we denote an external leg with ingoing momentum k_p simply by the label p, then we can make the association of Fig. 4.2.

Now, we have already established that BRST symmetries of the building blocks are encoded by permutations of their indices. What happens to the Mandelstam invariants under these permutations? We can check our known case to gain some intuition:

$$2QT_{1[23]} = -s_{123}T_{23}V^1 + s_{23}(T_{12}V^3 - T_{13}V^2 + T_{23}V^1). \tag{4.35}$$

The invariants are s_{123} and s_{23}. What kind of a diagram has these invariants? Fig. 4.3a does.



(a) Our known example (b) The general case

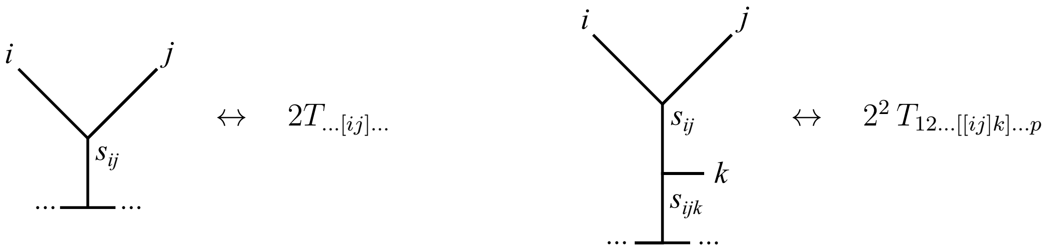
Figure 4.3: Antisymmetrizing two labels.

In general, we can antisymmetrize two indices by connecting the corresponding branches into a new vertex. If $T_{1\ldots ijk\ldots p} \to T_{1\ldots i[jk]\ldots p}$, this has the effect of changing $s_{1\ldots i} \to s_{jk}$ in the set of invariants, which matches up with the invariants of Fig. 4.3b. This can also be done recursively, with multiple antisymmetrizations forming extra branches on the new branch, as in Fig. 4.4b.

At this point, it seems like we have introduced an ambiguity into the notation. If branches are to be antisymmetric, then Figs. 4.2a and 4.3a are the same, except that the labels are permuted by $1 \leftrightarrow 3$. If we are to trust the diagrams, it seems that we must have

$$T_{123} = 2T_{3[21]} = T_{321} - T_{312}. \tag{4.36}$$

But this is precisely the rank-3 BRST symmetry (4.30)! The rank-2 symmetry (4.22) can also easily seen by considering Fig. 4.4a as either a tree or an antisymmetrizing branch,



(a) A single antisymmetrization

(b) The recursive behaviour

Figure 4.4: The branch: antisymmetrizing in general.

but this is a deeper property of the definition of the T_{ij} and as such is not multiplied by an overall factor of 2.

Every diagram can be interpreted as either a simple tree or a branched one. Equating the two yields the BRST symmetries of the T, or equivalently the BRST-exact combinations of the $\tilde{T}$. From this interpretation it is also clear that rank-p building blocks inherit the $p-2$ lower order symmetries in their first $p-1$ labels, such that we can find $R^{(\ell)}_{123\ldots p}$, $\ell = 1 \ldots p-1$. We can make these different forms explicit by exchanging the two arms of various branches, gaining a factor of -1 for each exchange. This is demonstrated for rank 4 in Fig. 4.5.

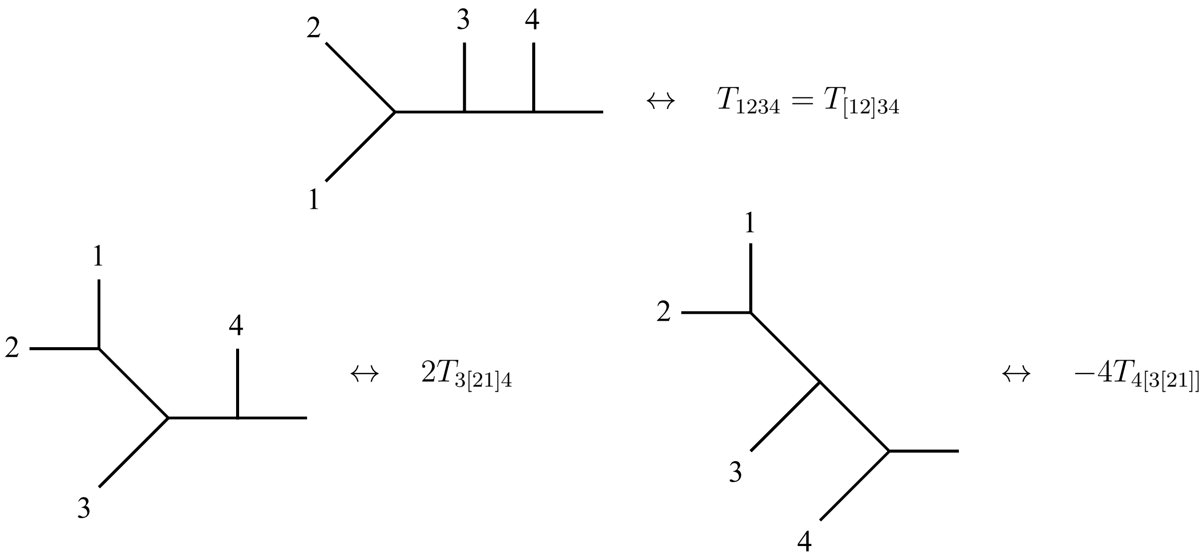


Figure 4.5: Various interpretations of the T_{1234} diagram. To get the bottom left diagram from the top, we exchange legs 1 and 2, and then exchange [21] and 3. For the bottom right diagram, we additionally exchange leg 4 and the unlabelled leg.

In general, at rank p, changing the order of indices around the diagram from $123\ldots p$ to $p\ldots 321$ results in an overall sign of $(-1)^{p-1}$, such that we always end up with order-p

symmetries of the form

$$T_{12\dots p} + (-1)^p\, T_{p(p-1)\dots 1} + \cdots = 0. \tag{4.37}$$

Lower order symmetries inherited from the previous steps as usual. This relative sign is most easily seen from diagrams of the form of Fig. 4.6. For example, at $p = 2$, the symmetry is $T_{12} + (-1)^2\, T_{21} = 0$. At $p = 3$ we have this same relation in the first two indices, $T_{123} + T_{213} = 0$, but the new highest rank symmetry takes the form $T_{123} - 2T_{3[21]} = T_{123} + (-1)^3\, T_{321} + \cdots = 0$.

We can make this even more concrete by considering the next available example, T_{1234}, and the diagrams in Fig. 4.5. We can interpret the diagram in four ways (the tree and three branched interpretations), leading to three BRST symmetries. We obtain the familiar rank-2 and rank-3 symmetries in the first 2 and 3 labels respectively, and also a new symmetry in all 4 labels:

$$T_{1234} + T_{[21]34} = T_{1234} + (-1)^2\, T_{2134} = 0 \tag{4.38a}$$

$$T_{1234} - 2T_{3[21]4} = T_{1234} + (-1)^3\, T_{3214} + T_{3124} = 0 \tag{4.38b}$$

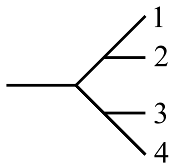
$$T_{1234} + 4T_{4[3[21]]} = T_{1234} + (-1)^4\, T_{4321} + T_{2143} + T_{3412} = 0. \tag{4.38c}$$

Equivalently, these are also the BRST exact combinations of $\tilde{T}_{1234}$, e.g.

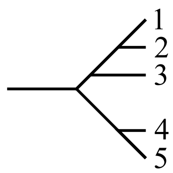
$$\tilde{T}_{1234} - \tilde{T}_{1243} + \tilde{T}_{3412} - \tilde{T}_{3421} \equiv Q R^{(3)}_{1234}. \tag{4.39}$$

Should we so desire, we can at this point use our OPEs and SYM field equations to find the explicit superfield representation of the R by brute force, secure in our knowledge that this combination of $\tilde{T}$ can be rewritten into the desired form.

For higher ranks, it is easier to see the highest order relation by putting the one open leg as close to the middle of the diagram as possible, and considering either the top or the bottom to be the antisymmetrization, as in Fig. 4.6. The lower order relations are simply inherited, so it is most convenient to make the highest order symmetry the most obvious.



$\leftrightarrow \quad 2T_{12[34]} = -2T_{43[21]}$



$\leftrightarrow \quad 2T_{123[45]} = 4T_{54[3[21]]}$

Figure 4.6: An easily generalizable way to find higher order BRST symmetries. The diagram can be both from top to bottom, and from bottom to top, with the second set of labels (consecutively) antisymmetrized in each case. The two interpretations have a relative sign of $(-1)^{p-1}$ since we must exchange the legs of $p-1$ branches to reverse the label ordering, in accordance with (4.37).

From this form of the diagrams we can deduce the general form of the BRST symmetry at rank p, with separate cases for even and odd p:

$$T_{123\dots r[r+1[\dots[p-2[(p-1),p]]\dots]]} + T_{p(p-1)\dots r+1[r[\dots[3[21]]\dots]]} = 0, \qquad p = 2r,\ r \in \mathbb{Z} \tag{4.40a}$$

$$T_{123\dots r+1[r+2[\dots[p-2[(p-1),p]]\dots]]} - 2T_{p(p-1)\dots r+2[r+1[\dots[3[21]]\dots]]} = 0, \qquad p = 2r+1. \tag{4.40b}$$

4.3 Berends-Giele Currents

We now define the so-called supersymmetric Berends-Giele (BG) currents[1] $M_{123\ldots p}$, the objects with which we can actually calculate amplitudes. As with previous superfields, these objects are defined recursively:

$$E_{12\ldots p} \equiv \sum_{j=1}^{p-1} M_{12\ldots j} M_{j+1\ldots p}, \qquad QM_{12\ldots p} \equiv E_{12\ldots p}, \qquad M_1 = V^1. \tag{4.41}$$

We see that as long as our SYM fields are on-shell, $E_1 = QV^1 = 0$.

As with T, we examine the first few ranks to gain some intuition. At rank 2, we find

$$QM_{12} = E_{12} = M_1 M_2 = V^1 V^2 = \frac{QL_{21}}{s_{12}} = \frac{QT_{12}}{s_{12}}, \tag{4.42}$$

as we recall from (4.32). We are avoiding explicitly exact terms, so we choose

$$M_{12} = \frac{T_{12}}{s_{12}}. \tag{4.43}$$

At rank 3, we get

$$QM_{123} = M_{12}M_3 + M_1 M_{23} = \frac{T_{12}V^3}{s_{12}} + \frac{V^1 T_{23}}{s_{23}}. \tag{4.44}$$

How do we find a solution for M here? We may begin by noticing that the combination $T_{12}V^3$ appears in the expression (4.33) for QT_{123}, repeated here for convenience:

$$QT_{123} = s_{123} T_{12} V^3 - s_{12}(T_{23}V^1 - T_{13}V^2 + T_{12}V^3). \tag{4.45}$$

This is promising, since we intuitively expect higher ranked M to involve higher rank T. Using the BRST symmetry $T_{23} = -T_{32}$ and noting that V^1 and T_{23} anticommute, we additionally note that

$$QT_{321} = s_{123} V^1 T_{23} + s_{23}(T_{12}V^3 - T_{13}V^2 + T_{23}V^1). \tag{4.46}$$

It is then clear that

$$\frac{1}{s_{12}} QT_{123} + \frac{1}{s_{23}} QT_{321} = s_{123}\left(\frac{T_{12}V^3}{s_{12}} + \frac{V^1 T_{23}}{s_{23}}\right), \tag{4.47}$$

and so we can choose

$$M_{123} = \frac{1}{s_{123}}\left(\frac{T_{123}}{s_{12}} + \frac{T_{321}}{s_{23}}\right). \tag{4.48}$$

[1] These are the supersymmetric analogues to colour ordered amplitudes with one off-shell leg considered by Berends and Giele in [19] for computing QCD amplitudes. See [14], [20], or Chapter 4 of [1] for more details.

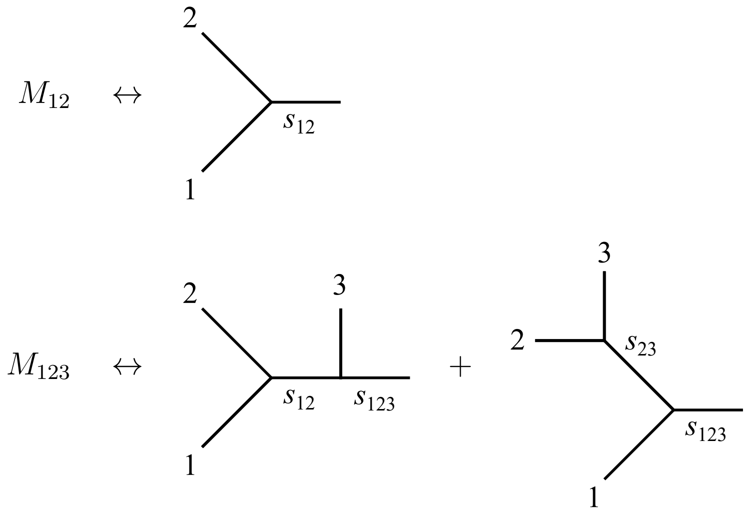


Figure 4.7: The diagrams of the T appearing in the expressions for the lowest order BG currents. The product of all s in each diagram is the denominator for the respective T in the sum.

This result may seem coincidental, but our graphical approach helps to elucidate the pattern. The diagrams associated with M_{12} and M_{123} are displayed in Fig. 4.7. In both cases we have a sum over a complete set of independent (i.e. not related by a BRST symmetry) rank-p diagrams, divided by all the Mandelstam invariants appearing in each diagram.

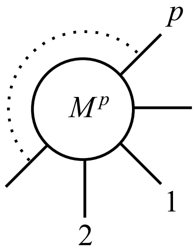
This pattern holds true in general: at each rank p, M is the sum of all independent rank-p building blocks, divided by all the Mandelstam invariants in their respective BRST variations. The relative signs between the diagrams are fixed by our definition (4.41) of M. Diagrammatically, we are summing over all tree diagrams respecting the label order (including the one unlabelled leg) of the M in question, and dividing by the product of all s in each diagram. The number[2] of such diagrams (rooted binary trees with unlabelled nodes and labelled leaves) is $(2p-2)!/(p!(p-1)!)$, as has been discussed in [7]. Comfortingly, this is also the number of ways of placing pairs of brackets in a word of p letters, i.e. the number of ways to antisymmetrize $T_{123\ldots p}$.

Since we are summing over all the possible ordered diagrams, we will diagrammatically represent the rank-p BG current as in Fig. 4.8. This also gives us an intuitive representation of E, the BRST variation of M. It is represented by the sum of all products of two split pieces of of the M diagram which respect the label order.

Now, since we have established clear symmetry properties of T, we would hope that these translate into symmetries of M. We begin by noticing that since $M_{12} \propto T_{12}$, we

[2]This is the $(p+1)^{\text{th}}$ Catalan number; these number form sequence A000108 in the On-Line Encyclopedia of Integer Sequences [21].

$$M_{123\ldots p} \quad \leftrightarrow$$



$$E_{123\ldots p} = QM_{123\ldots p} \quad \leftrightarrow$$

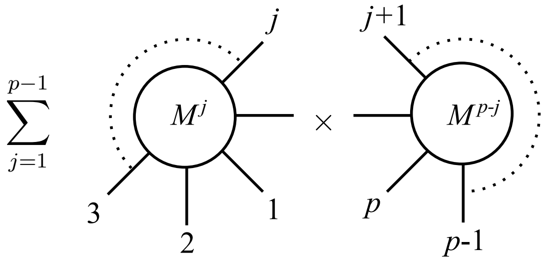


Figure 4.8: The rank-p BG current diagram is the sum of all cubic diagrams (corresponding to all antisymmetrizations of $T_{12\ldots p}$), divided by the product of all Mandelstam invariants appearing in each diagram. Its BRST variation is given by the sum of all possible splittings of this diagram respecting the ordering.

have

$$M_{12} + M_{21} = 0. \tag{4.49}$$

What about M_{123}? Since it contains both T_{123} and T_{321}, and since the s are insensitive to the order of their indices, we see immediately that

$$M_{123} - M_{321} = 0. \tag{4.50}$$

We can take this further without explicitly computing higher order M. Since we have taken care to make sure M contains no explicit exact terms, its BRST variation E must have the same symmetry properties as M itself. Examining, we find

$$E_{1234} = M_1M_{234} + M_{12}M_{34} + M_{123}M_4 = -M_{432}M_1 - M_{43}M_{21} - M_4M_{321} = -E_{4321}, \tag{4.51}$$

implying

$$M_{1234} + M_{4321} = 0. \tag{4.52}$$

It seems as though we can generalize to

$$M_{123\ldots p} + (-1)^p M_{p\ldots 21} = 0. \tag{4.53}$$

Let us assume that this is the case for rank p. Then at rank $p+1$,

$$\begin{aligned}
E_{12\ldots p+1} &= \sum_{j=1}^{p} M_{12\ldots j} M_{j+1\ldots p+1} \\
&= \sum_{j=1}^{p} (-1)(-1)^{(j-1)}(-1)^{(p+1)-(j+1)} M_{p+1\ldots j+1} M_{j\ldots 21} \\
&= (-1)^{p-2} \sum_{j=1}^{p} M_{p+1\ldots j+1} M_{j\ldots 21} = -(-1)^{p+1} E_{p+1\ldots 21},
\end{aligned} \tag{4.54}$$

so by induction we indeed have the reflection symmetry property (4.53). Furthermore, the sum of all cyclic permutations σ of the E labels vanish since the M anticommute:

$$\begin{aligned}
\sum_{\sigma} E_{\sigma(123\ldots n)} &= \sum_{\sigma} \sum_{p=1}^{n-1} M_{\sigma(12\ldots p} M_{p-1\ldots n)} \\
&= \frac{1}{2} \sum_{\sigma} \sum_{p=1}^{n-1} \left(M_{\sigma(12\ldots p} M_{p-1\ldots n)} + M_{\sigma(p-1\ldots n} M_{12\ldots p)} \right) = 0.
\end{aligned} \tag{4.55}$$

4.3.1 $\mathcal{A}_{\mathrm{YM}}$ from BG Currents

The symmetry properties of the $M_{123\ldots}$ bear a remarkable resemblance to the colour-ordered (S)YM subamplitude symmetries from chapter 2, which had a reflection symmetry (2.12) of the same form as (4.53). We do not prove this here as we maintain our focus on string theory, but it was shown in [14] and also discussed in [1] that $\mathcal{A}_{\mathrm{YM}}$ at N points can in fact be written as

$$\mathcal{A}_{\mathrm{YM}}(1,2,3,\ldots,N) = \langle E_{123\ldots N-1} V^N \rangle = \sum_{j=2}^{N-2} \langle M_{12\ldots j} M_{j+1\ldots N-1} V^N \rangle. \tag{4.56}$$

This equation gives us the subcyclic symmetry (2.13), and also the reflection symmetry since E and V are both fermionic. The cyclicity (2.11) is slightly more involved to show, but the basic idea is to use the BRST integration by parts procedure described section 3.3, combined with that $M_i V^j = M_{ij} = E_{ij}$ to absorb the seemingly special leg V^N into a cyclically symmetric sum over M's and E's. Finally, it is worth making a quick note about the seeming BRST triviality of the E appearing in the amplitude. Until now, we have been thinking of $E_{123\ldots j}$ as the BRST variation of $M_{123\ldots j}$. However, we have already shown that

$$M_{123\ldots j} \propto \frac{1}{s_{123\ldots j}}. \tag{4.57}$$

In the massless N-point case, $j = N-1$. Conservation of momentum requires $s_{123\ldots j} \propto (k^1 + k^2 + \cdots + k^{N-1})^2 = (k^N)^2 = 0$, and as such the object $M_{123\ldots N-1}$ cannot be constructed; it is forbidden by the kinematics. Thus $E_{123\ldots N-1}$ is not itself a BRST variation of any object at N points, but rather a full-fledged member of the cohomology of Q. Of course it is still closed, and we also have $QV = 0$ for on-shell SYM fields, so the whole amplitude is BRST invariant as required. The amplitude in terms of momenta and polarizations for gluons and gluinos can be obtained using the component expansion (2.9). Since this process can be very lengthy, a `FORM` package named `PSS` has been developed and published in [22] to aid in calculations.

In calculations to come, we will make use of this expression for $\mathcal{A}_{\text{YM}}$, but for our purposes it will be sufficient to consider it as a shorthand for the correlator (4.56). For completeness, we can diagrammatically represent this amplitude using our established notation and in keeping with [14] as in Fig. 4.9.

$$\mathcal{A}_{\text{YM}}(1, 2, \ldots, N) \quad \leftrightarrow \quad \sum_{j=1}^{N-2}$$

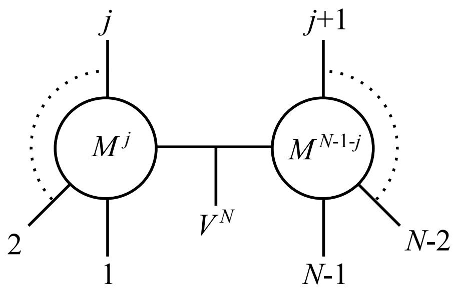


Figure 4.9: We can diagrammatically represent AYM by connecting the two off-shell legs of the $E_{12\ldots N-1}$ diagram with a V^N operator.

5 Tree-Level Amplitude Construction

Recall that the colour ordered tree-level N-point amplitude in the pure spinor formalism is given by

$$\mathcal{A}(1,2,\ldots,N) = \int \mathrm{d}z_2 \cdots \mathrm{d}z_{N-2} \left\langle V^1(0) V^{N-1}(1) V^N(\infty) U^2(z_2) U^3(z_3) \cdots U^{N-2}(z_{N-2}) \right\rangle , \tag{5.1}$$

where the integration region $0 = z_1 < z_2 < z_3 < \ldots < z_{N-2} < z_{N-1} = 1$ determines the colour ordering $123\ldots N$. It is immediately clear that the 3-point amplitude is given by

$$\mathcal{A}(1,2,3) = \left\langle V^1 V^2 V^3 \right\rangle , \tag{5.2}$$

which can be written in terms of momenta and polarizations using the component expansion (2.9) and picking out the relevant terms using the correlator. The 4-point case is slightly more involved, but 5 points is where the real subtlety of the task starts coming into play.

In the following, we will demonstrate our techniques on the 4-point case, extend to 5 points, and then present the general case. First we will introduce some condensed notation to maximize our efficiency in computations, and then we will calculate the desired amplitudes in detail. The general procedure is as follows: first we write the correlator in terms of a combination of various L by taking various OPEs between the unintegrated vertex operator(s), V^1, and V^{N-1}. Next, we rewrite these in terms of building blocks, and then in terms of BG currents. Finally, we perform an integration by parts procedure to factor all explicit z and s dependence, and combine all the correlators into one compact expression.

5.1 Notation Condensation

As is often necessary when computing amplitudes, we will condense our notation to avoid clutter. We will call the amplitude in the canonical colour ordering $123\ldots N$ $\mathcal{A}_N \equiv \mathcal{A}(1,2,\ldots,N)$.

We will abuse our existing notation somewhat by implicitly factoring out the X^m dependence of the vertex operators. From the SYM component expansions, and noting that both V and each term in U contains exactly one SYM field, we see that the vertex operators can be written as

$$V^j(X(z_j),\theta(z_j)) = e^{k^j\cdot X}V^j(\theta), \qquad U^j(X,\theta) = e^{k^j\cdot X}U^j(\theta). \tag{5.3}$$

As we know from open bosonic string theory (see e.g. chapter 6 of [8] for an overview), which has the same OPE between two X as in our case,

$$\left\langle \prod_{j=1}^{N} e^{i(k^j\cdot X(z_j))} \right\rangle \propto \delta^D\Big(\sum_j k^j\Big)\prod_{j=1}^{N}\prod_{i<j}|z_{ij}|^{s_{ij}}. \tag{5.4}$$

In our case, we change the sign of the s_{ij} in the power of $|z_{ij}|$ because we are absorbing the factor of i into our momenta[1], such that $\partial_m \leftrightarrow k_m$ instead of $\partial_m \leftrightarrow ik_m$.

So in our case the X dependent factors carry explicit z dependence, which we are interested in. As such, we will often factor this dependence out (and drop the overall momentum conservation Dirac delta, which is implied). However, we will continue to just call the relevant vertex operators $V^i(z_i)$ and $U^j(z_j)$, in keeping with the notation of [1]. That this factorization has taken place will be implied by the presence of the explicit product of z_{ij} in the integrals:

$$\mathcal{A}_N = \prod_{\ell=2}^{N-2}\int \mathrm{d}z_\ell \prod_{i<j}|z_{ij}|^{-s_{ij}}\langle V^1V^{N-1}V^NU^2U^3\cdots U^{N-2}\rangle. \tag{5.5}$$

Where the meaning is clear, we will replace the cumbersome integration factor simply by an integral sign:

$$\int\langle V^1V^{N-1}V^NU^2U^3\cdots U^{N-2}\rangle \equiv \prod_{\ell=2}^{N-2}\int \mathrm{d}z_\ell \prod_{i<j}|z_{ij}|^{-s_{ij}}\langle V^1V^{N-1}V^NU^2U^3\cdots U^{N-2}\rangle. \tag{5.6}$$

Now, since the vertex operator V^N is inserted infinitely far away from the others, it will often play no role in the computation of the correlators. To simplify the notation and prevent having to carry non-participatory factors of V^N throughout entire calculations, we will denote this final vertex operator with a subscript on the correlator brackets:

$$\langle V^1V^{N-1}U^2U^3\cdots U^{N-2}\rangle_N \equiv \langle V^1V^{N-1}U^2U^3\cdots U^{N-2}\,V^N(z_N=\infty)\rangle. \tag{5.7}$$

Because $QV=0$, BRST integration by parts functions in the same way for $\langle\rangle_N$ correlators as it would if the V^N was included explicitly.

Putting everything together, we can more conveniently write the full N-point amplitude as

$$\mathcal{A}_N = \int\langle V^1U^2\cdots U^{N-2}V^{N-1}\rangle_N. \tag{5.8}$$

[1] This difference was pointed out in [5].

5.2 Manipulating the Kinematic Integrals

The various integrals involved in constructing the tree-level amplitude have been examined in significant detail in [7]; here we will extract only what is needed for our computation. We have many integrals containing the product of $|z_{ij}|^{-s_{ij}}$ for $i < j$. In the canonical colour ordering, $z_i < z_{i+1}$, so $|z_{ij}| = -z_{ij} = z_{ji}$. Let us assume this colour ordering for the remainder of the section.

The key to manipulating the integrals with kinematic factors is an integration by parts procedure, where we assume vanishing boundary terms. We first note that

$$\begin{aligned} 0 &= -\int \prod_{\ell=2}^{N-2} \mathrm{d}z_\ell \, \frac{\partial}{\partial z_k} \prod_{1\le i<j\le N-1} |z_{ij}|^{-s_{ij}} = -\int \prod_{\ell=2}^{N-2} \mathrm{d}z_\ell \, \frac{\partial}{\partial z_k} \prod_{i<j} (z_{ji})^{-s_{ji}} \\ &= \int \prod_{\ell=2}^{N-2} \mathrm{d}z_\ell \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \left(\sum_{\substack{m=1\\ m\neq k}}^{N-1} \frac{s_{km}}{z_{km}} \right) = \int \prod_{\ell=2}^{N-2} \mathrm{d}z_\ell \, \prod_{i<j} |z_{ij}|^{-s_{ij}} \left(\sum_{m<k} \frac{s_{km}}{z_{km}} - \sum_{m>k} \frac{s_{mk}}{z_{mk}} \right). \end{aligned} \tag{5.9}$$

This is already a useful expression for swapping out kinematic factors in sums of single pole integrals, but we would like to also include arbitrary z in the denominator. Let us require that all additional factors of z be ordered z_{mn} for $m > n$, to match the final form of the above equation. Then we can add in these factors simply by shifting the relevant s_{ji}:

$$\prod_{i<j} |z_{ij}|^{-s_{ij}} \frac{1}{(z_{ji})^{n_{ji}}} = \prod_{i<j} (z_{ji})^{-s_{ji}-n_{ji}}, \tag{5.10}$$

so

$$\begin{aligned} &-\int \prod_{\ell=2}^{N-2} \mathrm{d}z_\ell \, \frac{\partial}{\partial z_k} \prod_{i<j} \frac{|z_{ji}|^{-s_{ji}}}{(z_{ji})^{n_{ji}}} \\ &\qquad = \int \prod_{\ell=2}^{N-2} \mathrm{d}z_\ell \, \prod_{i<j} \frac{|z_{ji}|^{-s_{ji}}}{(z_{ji})^{n_{ji}}} \left(\sum_{m<k} \frac{s_{km}+n_{km}}{z_{km}} - \sum_{m>k} \frac{s_{mk}+n_{mk}}{z_{mk}} \right) = 0. \end{aligned} \tag{5.11}$$

We can write this in a simpler form assuming that all n_{mk} and n_{km} are zero, i.e. the differentiation variable appears only in the Veneziano factor $\prod |z_{ij}|^{-s_{ij}}$:

$$\int \prod_{\ell=2}^{N-2} \mathrm{d}z_\ell \, \frac{\partial}{\partial z_k} \prod_{i<j} \frac{|z_{ji}|^{-s_{ji}}}{(z_{ji})^{n_{ji}}} = \int \prod_{\ell=2}^{N-2} \mathrm{d}z_\ell \, \prod_{i<j} \frac{|z_{ji}|^{-s_{ji}}}{(z_{ji})^{n_{ji}}} \left(\sum_{\substack{m=1\\ m\neq k}}^{N-1} \frac{s_{km}}{z_{km}} \right), \quad n_{mk} = n_{km} = 0 \;\forall\; m. \tag{5.12}$$

Let's apply these formulae to the 5-point amplitude. For example, the L_{2331} term in $\langle V^1 U^2 U^3 V^4 \rangle$ (see (5.31)) has a denominator of the form $1/z_{23}z_{31} = -1/z_{32}z_{31}$. Choosing

$n_{31} = 1$ and $k = 2$ allows us to use the simple form (5.12), giving the result

$$-L_{2331}V^4 \int \mathrm{d}z_2\mathrm{d}z_3 \prod_{i<j} |z_{ij}|^{-s_{ij}} \frac{1}{z_{31}z_{32}} = -\frac{L_{2331}V^4}{s_{32}} \int \mathrm{d}z_2\mathrm{d}z_3 \prod_{i<j} |z_{ij}|^{-s_{ij}} \frac{1}{z_{31}} \left(\frac{s_{21}}{z_{21}} + \frac{s_{24}}{z_{24}} \right). \tag{5.13}$$

The more general case can be exploited to rewrite the double pole term of the same correlator, which is proportional to $(1 + s_{32})/(z_{32})^2$. Choosing $n_{32} = 1$ and $k = 3$, we can rewrite:

$$\int \mathrm{d}z_2\mathrm{d}z_3 \prod_{i<j} |z_{ij}|^{-s_{ij}} \frac{1}{z_{32}} \frac{s_{32}+1}{z_{32}} = -\int \mathrm{d}z_2\mathrm{d}z_3 \prod_{i<j} |z_{ij}|^{-s_{ij}} \frac{1}{z_{32}} \left(\frac{s_{31}}{z_{31}} - \frac{s_{43}}{z_{43}} \right). \tag{5.14}$$

Integrating by parts combined with the use of the easily proven partial fraction identity

$$\frac{1}{z_{ji}z_{ki}} + \frac{1}{z_{jk}z_{ji}} = \frac{1}{z_{jk}z_{ki}} \tag{5.15}$$

gives us significant flexibility in rearranging the integrals appearing in amplitudes.

5.3 Correlators from Building Blocks

The correlator appearing in the four-point amplitude is $\langle V^1U^2V^2V^4\rangle$. We have not written down the integrals or positions of the vertex operators, but we use the $SL(2,\mathbb{R})$ invariance of the worldsheet to put V^4 (or V^N in general) infinitely far away, so we can neglect any OPEs involving this operator. However, we do end up with terms involving OPEs between the rest of the operators. Recalling that the V are fermionic and U is bosonic, we have

$$\langle V^1(z_1)U^2(z_2)V^3(z_3)V^4(z_4)\rangle = \langle V^1U^2V^3V^4\rangle \to \left\langle \frac{L_{21}V^3V^4}{z_{21}} \right\rangle + \left\langle \frac{V^1L_{23}V^4}{z_{23}} \right\rangle. \tag{5.16}$$

Now, to move on to using the building blocks, we note that T_{12} differs from L_{21} only by the BRST exact term $\frac{1}{2}QD_{12} \equiv \frac{1}{2}Q(A^1 \cdot A^2)$. As such, we can write

$$\langle L_{21}V^3V^4\rangle = \langle T_{12}V^3V^4\rangle - \frac{1}{2}\langle (QD_{12})V^3V^4\rangle. \tag{5.17}$$

The BRST operator Q acts like a differential operator, and $QV = 0$. Since BRST exact terms decouple, we get

$$\langle (QD_{12})V^3V^4\rangle = \langle Q(D_{12}V^3V^4)\rangle = 0, \tag{5.18}$$

and doing the same for the L_{23} term gives

$$\langle V^1U^2V^3V^4\rangle \to \left\langle \frac{T_{12}V^3V^4}{z_{21}} \right\rangle + \left\langle \frac{V^1T_{32}V^4}{z_{23}} \right\rangle, \tag{5.19}$$

so we have achieved our goal fairly straightforwardly.

At 5 points, the situation becomes messier. This is mostly due to terms arising from the OPE of two integrated vertex operators. From (A.3), we can write this as

$$
\begin{aligned}
\lim_{z_2\to z_3} U^2(z_2)U^3(z_3) \to \frac{1}{z_{23}} &\big[(W^3\gamma_m W^2)(\Pi^m + k_n^{23}N^{nm}) - N^{mn}\mathcal{F}^3{}_m{}^p\mathcal{F}^2_{np} \\
&\qquad + [A^2_m(W^3\gamma^m\partial\theta) + (k^2\cdot\Pi)(A^3W^2) + (A^3\cdot k^2)U^2 - (2\leftrightarrow 3)]\big] \\
&+ \frac{1+s_{23}}{z_{23}^3}\big[2(A^{(3}W^{2)}) - (A^3\cdot A^2)\big] \\
&\equiv \frac{1}{z_{23}}Y^{23} + \frac{1+s_{23}}{z_{23}^2}P^{23}.
\end{aligned}
\tag{5.20}
$$

We have a set of single poles ($\propto z_{ij}^{-1}$) and additionally a set of double poles ($\propto z_{ij}^{-2}$).

Let us first focus on the single poles. We expect that the physics should not be affected by the order in which we carry out OPEs between the various vertex operators. For example, the sequence beginning with U^2,

$$
V^1(z_1)U^2(z_2)U^3(z_3) \to \frac{(V^1U^2)(z_1)\,U^3(z_3)}{z_{21}} + \frac{V^1(z_1)\,(U^2U^3)(z_3)}{z_{23}},
\tag{5.21}
$$

should yield equivalent results to the sequence beginning with U^3,

$$
V^1(z_1)U^2(z_2)U^3(z_3) \to \frac{(V^1U^3)(z_1)\,U^2(z_2)}{z_{31}} + \frac{V^1(z_1)\,(U^3U^2)(z_2)}{z_{32}}.
\tag{5.22}
$$

Now, of course, we can express these OPEs in terms of our known L. Hearkening back to the generalized notation introduced in (4.2), we write

$$
(5.21) \to \frac{L_{2131}}{z_{21}z_{31}} + \frac{L_{2331}}{z_{23}z_{31}}
\tag{5.23}
$$

and

$$
(5.22) \to \frac{L_{3121}}{z_{31}z_{21}} + \frac{L_{3221}}{z_{32}z_{21}}.
\tag{5.24}
$$

Setting these two expressions equal to each other will give us a relation between the various L. But first, we note that we can Taylor expand the UU single poles

$$
\frac{Y^{32}(z_2)}{z_{32}} = -\frac{Y^{32}(z_3)}{z_{23}} + \text{regular},
\tag{5.25}
$$

implying that

$$
L_{3221} = -L_{2331}.
\tag{5.26}
$$

Using this and equating our two expressions, we find that

$$
\frac{L_{3121} - L_{2131}}{z_{21}z_{31}} = L_{2331}\left(\frac{1}{z_{23}z_{31}} - \frac{1}{z_{23}z_{21}}\right) = \frac{L_{2331}}{z_{21}z_{31}},
\tag{5.27}
$$

where after the second equality we used the partial fraction relation (5.15). So

$$L_{2331} = L_{3121} - L_{2131} \equiv 2\,L_{[31,21]}. \tag{5.28}$$

This identity can also be checked explicitly from the superfield expressions of the L, as has been done in appendix A of [5]. This result generalizes to higher ranks:

$$z_{23}z_{34}z_{41} \lim_{z_2\to z_3\to z_4\to z_1} V^1(z_1)U^2(z_2)U^3(z_3)U^4(z_4) \to L_{233441} = 2^2\,L_{[41,[31,21]]} \tag{5.29}$$

$$L_{233445\ldots(p-1)pp1} = 2^{p-2}\,L_{[p1,[(p-1)1,[\ldots,[31,21]\ldots]]]}. \tag{5.30}$$

At this point we can include the double poles of U^2U^3 in our considerations by writing the full correlator including V^4 (and V^5). Continuing to use the generalized L, we can write

$$\begin{aligned}\langle V^1U^2U^3V^4\rangle_5 \to{}& \left\langle \frac{2L_{[31,21]}V^4}{z_{23}z_{31}} + \frac{L_{2131}V^4}{z_{21}z_{31}} + \frac{L_{21}L_{34}}{z_{21}z_{34}} + \frac{1}{2}\frac{1+s_{23}}{z_{23}^2}P^{23}V^1V^4 - (1\leftrightarrow 4)\right\rangle_5 \\ ={}& \left\langle \frac{L_{2131}V^4}{z_{12}z_{23}} + \frac{L_{21}L_{34}}{z_{12}z_{43}} + \frac{V^1L_{3424}}{z_{43}z_{32}} + \frac{1}{2}\frac{1+s_{23}}{z_{23}^2}P^{23}V^1V^4 + (2\leftrightarrow 3)\right\rangle_5 .\end{aligned} \tag{5.31}$$

In the second line we used the partial fraction relation to regroup the rank-4 L terms into a 2-3 symmetric form, or equivalently considered only the OPE orders beginning with z_2 and then took care of those beginning with z_3 in the $(2\leftrightarrow 3)$. We can rewrite the product of rank-2 L in terms of building blocks fairly simply:

$$\begin{aligned}\langle L_{21}L_{34}\rangle_5 &= \langle (T_{12} - \frac{1}{2}QD_{12})(T_{43} - \frac{1}{2}QD_{34})\rangle_5 \\ &= \langle T_{12}T_{43}\rangle_5 - \frac{1}{2}\langle T_{12}(QD_{34})\rangle_5 - \frac{1}{2}\langle (QD_{12})T_{43}\rangle_5 \\ &= \langle T_{12}T_{43}\rangle_5 - \frac{1}{2}s_{12}\langle V^1V^2D_{34}\rangle_5 + \frac{1}{2}s_{34}\langle D_{12}V^4V^3\rangle_5.\end{aligned} \tag{5.32}$$

Here we BRST integrated by parts, used the T transformation property (4.31), and used our notation $D_{ij} = A^i\cdot A^j$. Note that we ended up with terms of opposite sign due to the opposite ordering of the T and D in the two remainder terms[2] (see (3.41)). To rewrite the other terms, we recall that

$$\tilde{T}_{123} = L_{2131} + \frac{1}{2}s_{12}(D_{13}V^2 - D_{23}V^1) + \frac{1}{2}(s_{13}+s_{23})D_{12}V^3, \tag{5.33}$$

and so

$$\tilde{T}_{432} = L_{3424} + \frac{1}{2}s_{43}(D_{42}V^3 - D_{32}V^4) + \frac{1}{2}(s_{42}+s_{32})D_{43}V^2. \tag{5.34}$$

[2]Thank you to Joe Minahan and Thales Azevedo for discussions on this point.

Since the rank-3 L only appear multiplied by unintegrated vertex operators, and since $\tilde{T}$ and T differ only by BRST exact terms, we can make the replacement $L \to \tilde{T}$ and then immediately replace $\tilde{T} \to T$. We get

$$\langle V^1 U^2 U^3 V^4 \rangle_5 \to \left\langle \frac{T_{123} V^4}{z_{12} z_{23}} + \frac{T_{12} T_{43}}{z_{12} z_{43}} + \frac{V^1 T_{432}}{z_{43} z_{32}} + (2 \leftrightarrow 3) \right\rangle_5 + \langle \mathcal{R}_5 \rangle_5 \,, \tag{5.35}$$

where we defined $\mathcal{R}_5$ to be the remainder terms resulting from the $L \to \tilde{T}$ exchange, plus the double pole term:

$$\begin{aligned}
\mathcal{R}_5 \equiv -\frac{1}{2} \Bigg[& \frac{s_{12}(D_{13} V^2 - D_{23} V^1) V^4}{z_{12} z_{23}} + \frac{(s_{13} + s_{23}) D_{12} V^3 V^4}{z_{12} z_{23}} + \frac{s_{12} V^1 V^2 D_{34}}{z_{12} z_{43}} \\
& - \frac{s_{34} D_{12} V^4 V^3}{z_{12} z_{43}} + \frac{s_{43} V^1 (D_{42} V^3 - D_{32} V^4)}{z_{43} z_{32}} + \frac{V^1 (s_{42} + s_{32}) D_{43} V^2}{z_{43} z_{32}} \\
& - \frac{1 + s_{23}}{z_{23}^2} P^{23} V^1 V^4 + (2 \leftrightarrow 3) \Bigg] \\
= -\frac{1}{2} \Bigg[& D_{12} V^3 V^4 \left(\frac{s_{13}}{z_{13} z_{32}} + \frac{s_{13} + s_{23}}{z_{12} z_{23}} + \frac{s_{43}}{z_{12} z_{43}} \right) \\
& - D_{42} V^3 V^1 \left(\frac{s_{13}}{z_{13} z_{42}} + \frac{s_{43}}{z_{43} z_{32}} + \frac{s_{43} + s_{23}}{z_{42} z_{23}} \right) \\
& + \frac{1}{2} D_{23} V^1 V^4 \left(- \frac{s_{12}}{z_{12} z_{23}} - \frac{s_{43}}{z_{43} z_{32}} - \frac{s_{13}}{z_{13} z_{32}} - \frac{s_{42}}{z_{42} z_{23}} \right) \\
& - \frac{1 + s_{23}}{z_{23}^2} P^{23} V^1 V^4 + (2 \leftrightarrow 3) \Bigg] \,.
\end{aligned} \tag{5.36}$$

Note that in the last line, the factor of $1/2$ on the D_{23} term appeared to compensate for symmetrizing the term with respect to 2-3.

Now, since our goal is to write $\langle V^1 U^2 U^3 V^4 \rangle_5$ in terms of building blocks, we would like this remainder to vanish. Of course, in the end it is not the correlator itself we would like to rewrite, but the amplitude. So we are actually trying to ascertain whether

$$\int \mathrm{d}z_2 \mathrm{d}z_3 \prod_{i<j} |z_{ij}|^{-s_{ij}} \langle \mathcal{R}_5 \rangle_5 \stackrel{?}{=} 0. \tag{5.37}$$

This leaves us free to use our integral machinery from the previous section. Beginning with the first term, we can ignore the D and V factors since they are not integrated over. We are left with only the integral of the various s/z terms, which is promising. Now, using the partial fraction relation and the simple integration by parts (5.12) with $k = 3$,

$$\begin{aligned}
\int \mathrm{d}z_2 \mathrm{d}z_3 \prod_{i<j} |z_{ij}|^{-s_{ij}} \left(\frac{s_{13}}{z_{13} z_{32}} + \frac{s_{13} + s_{23}}{z_{12} z_{23}} \right) &= \int \left(s_{12} \left(\frac{1}{z_{13} z_{32}} + \frac{1}{z_{12} z_{23}} \right) + \frac{s_{23}}{z_{12} z_{23}} \right) \\
&= \int \frac{1}{z_{21}} \left(\frac{s_{31}}{z_{31}} + \frac{s_{32}}{z_{32}} \right) = - \int \frac{s_{43}}{z_{12} z_{43}}.
\end{aligned} \tag{5.38}$$

So the D_{12} term vanishes! The second term, proportional to D_{42}, has the exact same form as the first, just with $1 \to 4$ and the opposite sign. So it too vanishes, leaving us only with the D_{23} and double pole terms. To treat the D_{23} term we use the full integration by parts formula (5.11) with both $k = 3$ and $k = 2$:

$$\begin{aligned}\int \left(-\frac{s_{12}}{z_{12}z_{23}} - \frac{s_{43}}{z_{43}z_{32}} - \frac{s_{13}}{z_{13}z_{32}} - \frac{s_{42}}{z_{42}z_{23}}\right) &= \int \left(\frac{1}{z_{32}}\left(\frac{s_{31}}{z_{31}} - \frac{s_{43}}{z_{43}}\right) - \frac{1}{z_{32}}\left(\frac{s_{21}}{z_{21}} - \frac{s_{42}}{z_{42}}\right)\right) \\ &= -2\int \frac{1+s_{32}}{(z_{32})^2}. \end{aligned} \tag{5.39}$$

Now, $P^{23} = 2(A^{(2}W^{3)}) - D_{23}$, so we have

$$\begin{aligned}\int \mathrm{d}z_2\mathrm{d}z_3 \prod_{i<j} |z_{ij}|^{-s_{ij}} \langle \mathcal{R}_5 \rangle_5 &= \frac{1}{2}\int \frac{1+s_{23}}{(z_{23})^2}(D_{23}V^1V^4 + P^{23}V^1V^4) \\ &= \int \frac{1+s_{23}}{(z_{23})^2}(A^{(2}W^{3)})V^1V^4. \end{aligned} \tag{5.40}$$

This term is in fact exactly cancelled by the BRST exact term which we dropped from the rank-2 L and its descendent terms which would have appeared in the rank-3 L had we not dropped it. We did not include it so as to preserve the useful transformation properties of the L and T. However, carrying these terms along separately, we find that they produce terms of the form (A^iW^j) multiplied by sums of $s_{k\ell}$ which vanish as a result of momentum conservation. The details of this cancellation are given in [5]. Of course, we could already have known that these double poles would cancel, since their presence would indicate tachyons in the spectrum, which we do not have in superstring theory. So we are finally left with the confirmation of (5.37), that indeed

$$\int \mathrm{d}z_2\mathrm{d}z_3 \prod_{i<j} |z_{ij}|^{-s_{ij}} \langle \mathcal{R}_5 \rangle_5 = 0. \tag{5.41}$$

We have confirmed that we can rewrite

$$\begin{aligned}\mathcal{A}_5 &= \int \langle V^1U^2U^3V^4\rangle_5 \\ &\to \int \left\langle \frac{L_{2131}V^4}{z_{12}z_{23}} + \frac{L_{21}L_{34}}{z_{12}z_{43}} + \frac{V^1L_{3424}}{z_{43}z_{32}} + \frac{1}{2}\frac{1+s_{23}}{z_{23}^2}P^{23}V^1V^4 + (2 \leftrightarrow 3)\right\rangle_5 \\ &= \int \left\langle \frac{T_{123}V^4}{z_{12}z_{23}} + \frac{T_{12}T_{43}}{z_{12}z_{43}} + \frac{V^1T_{432}}{z_{43}z_{32}} + (2 \leftrightarrow 3)\right\rangle_5. \end{aligned} \tag{5.42}$$

Taking the natural extension $T_i = L_i = V^i$, we can write this result as

$$\mathcal{A}_5 = \int \sum_{p=1}^{3} \left\langle \frac{T_{12\ldots p}T_{43\ldots p+1}}{z_{12\ldots p}z_{43\ldots p+1}} + \mathcal{P}(2,3)\right\rangle_5, \tag{5.43}$$

where $\mathcal{P}(2,3)$ is denotes terms with the other permutation of $(2,3)$, and $z_{123\ldots p}$ is the consecutive pairwise differences of z with the order of the labels:

$$z_{ijk\ell\ldots MN} \equiv z_{ij} z_{jk} z_{k\ell} \cdots z_{MN}. \tag{5.44}$$

This form is unnecessarily cumbersome at 5 points, but has the obvious generalization

$$\mathcal{A}_N = \int \langle V^1 U^2 U^3 \cdots U^{N-2} V^{N-1} \rangle_N = \int \sum_{p=1}^{N-2} \left\langle \frac{T_{12\ldots p} T_{N-1,N-2,\ldots,p+1}}{z_{12\ldots p}\, z_{N-1,N-2,\ldots,p+1}} + \mathcal{P}(2,\ldots,N-2) \right\rangle_N . \tag{5.45}$$

The general result is the product of a similar process: we take all the relevant OPE orders, rewrite in terms of building blocks, and the remainder is cancelled by the higher order poles resulting from OPEs between pairs of integrated vertex operators.

5.4 Substituting BG Currents and $\mathcal{A}_{\mathrm{YM}}$

At this point we have our amplitude as a sum over terms of the form $\langle TTV \rangle$. Recalling section 4.3.1, the YM subamplitude is a sum over terms of the form $\langle MMV \rangle$. Since the M are defined in terms of T and various Mandelstam invariants, it is reasonable to expect that we can simplify further by rewriting our current expression in terms of M and eventually $\mathcal{A}_{\mathrm{YM}}$. This is also a promising form for the transfer of knowledge between SYM and string theory.

We will now work at the level of the full amplitude, not just the correlator, as we will be making use of the integral machinery of section 5.2. As we have come to expect, the 4-point case is very simple. We have seen already in (5.19) that only the rank-2 building blocks appear in the amplitude. Recalling section 4.3, we can simply write $V^i = M_i$ and $T_{ij} = s_{ij} M_{ij}$, so we get

$$\mathcal{A}_4 = \int \mathrm{d}z_2 \prod_{i<j} |z_{ij}|^{-s_{ij}} \left\langle \frac{T_{12} V^3 V^4}{z_{21}} + \frac{V^1 T_{23} V^4}{z_{32}} \right\rangle = \int \left\langle \frac{s_{21}}{z_{21}} M_{12} M_3 V^4 + \frac{s_{32}}{z_{32}} M_1 M_{23} V^4 \right\rangle . \tag{5.46}$$

At 4 points,

$$\int \mathrm{d}z_2 \prod_{i<j} |z_{ij}|^{-s_{ij}} \frac{s_{32}}{z_{32}} = \int \mathrm{d}z_2 \prod_{i<j} |z_{ij}|^{-s_{ij}} \frac{s_{21}}{z_{21}}, \tag{5.47}$$

so

$$\begin{aligned} \mathcal{A}_4 &= \int \frac{s_{21}}{z_{21}} \left\langle (M_{12} M_3 + M_1 M_{23}) V^4 \right\rangle = \left\langle E_{123} V^4 \right\rangle \int \frac{s_{21}}{z_{21}} \\ &= \mathcal{A}_{\mathrm{YM}}(1,2,3,4) \int \mathrm{d}z_2 \prod_{i<j} |z_{ij}|^{-s_{ij}} \frac{s_{21}}{z_{21}}. \end{aligned} \tag{5.48}$$

Again the 5-point case is trickier. For ranks higher than 2, the BG currents are written in terms of several independent T (a complete set of them at the given rank, in fact), so there is not a simple one-to-one correspondence like in the 4-point case. Fortunately, the permutation sum allows for a rewriting. Using the partial fraction relation, the BRST symmetries of T_{ijk} (4.30), and the expression of M_{ijk} in terms of T (4.48), one can check that

$$\frac{T_{123}}{z_{12}z_{23}} + \mathcal{P}(2,3) = \frac{s_{12}}{z_{12}}\left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}}\right) M_{123} + \mathcal{P}(2,3). \tag{5.49}$$

One way to prove this is by multiplying the left side by $1 = (s_{12} + s_{23} + s_{13})/s_{123}$, grouping terms using $T_{321} = T_{123} - T_{132}$, adding $0 = s_{12}s_{13}(T_{321} + T_{231})/z_{12}z_{13}s_{23}$, and finally substituting in the expression for M. Note that this holds only when including the permutation sum!

Combining with our expression for the rank-2 M, we have

$$\begin{aligned}
\mathcal{A}_5 &= \int \mathrm{d}z_2\mathrm{d}z_3 \prod_{i<j} |z_{ij}|^{-s_{ij}} \left\langle \frac{T_{123}V^4V^5}{z_{12}z_{23}} + \frac{T_{12}T_{43}V^5}{z_{12}z_{43}} + \frac{V^1T_{432}V^5}{z_{43}z_{32}} + \mathcal{P}(2,3) \right\rangle \\
&= \int \left\langle \frac{s_{12}}{z_{12}}\left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}}\right) M_{123}V^4 + \frac{s_{12}s_{43}}{z_{12}z_{43}} M_{12}M_{43} \right. \\
&\qquad\qquad \left. + \frac{s_{43}}{z_{43}}\left(\frac{s_{42}}{z_{42}} + \frac{s_{32}}{z_{32}}\right) V^1 M_{432} + \mathcal{P}(2,3) \right\rangle_5 \\
&= \int \left\langle \frac{s_{12}}{z_{12}}\frac{s_{34}}{z_{34}} M_{123}V^4 + \frac{s_{12}s_{43}}{z_{12}z_{43}} M_{12}M_{43} + \frac{s_{43}}{z_{43}}\frac{s_{21}}{z_{21}} V^1 M_{432} + \mathcal{P}(2,3) \right\rangle_5 \\
&= \int \left\langle \frac{s_{12}s_{34}}{z_{12}z_{34}} \left(M_{123}V^4 - M_{12}M_{43} + V^1 M_{432}\right) + \mathcal{P}(2,3) \right\rangle_5 .
\end{aligned} \tag{5.50}$$

After integrating by parts, we find that all the terms within a given permutation end up over a common denominator! Furthermore, the resulting combination (between round brackets above) looks suspiciously like a Yang-Mills amplitude. Recall that the BG currents have the reflection symmetry (4.53) in their labels, resulting in an overall sign of $(-1)^{p-1}$ at rank p. This allows us to finally rewrite $\mathcal{A}_5$ into the desired form:

$$\langle M_{123}M_4 + M_{12}M_{34} + M_1M_{234} \rangle_5 = \left\langle E_{1234}V^5 \right\rangle = \mathcal{A}_{\mathrm{YM}}(1,2,3,4,5) \tag{5.51}$$

and so

$$\mathcal{A}(1,2,3,4,5) = \mathcal{A}_{\mathrm{YM}}(1,2,3,4,5) \int \mathrm{d}z_2\mathrm{d}z_3 \prod_{i<j} |z_{ij}|^{-s_{ij}} \frac{s_{12}s_{34}}{z_{12}z_{34}} \; + \; \mathcal{P}(2,3). \tag{5.52}$$

It is tempting at this point to generalize to $\mathcal{A}_{\mathrm{YM},N} \int s_{12}s_{23}s_{34}\cdots/z_{12}z_{23}z_{34}\cdots$, but the explicit analysis of the 6-point amplitude in [1, 6] shows that this would be premature. The general case requires a more careful treatment of the BG currents and the integration by parts.

Higher rank BG currents have been explicitly computed in the appendix of [1]. Using these, or applying the diagrammatic method, we can check that

$$\frac{T_{1234}}{z_{1234}} + \mathcal{P}(2,3,4) = \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \left(\frac{s_{14}}{z_{14}} + \frac{s_{24}}{z_{24}} + \frac{s_{34}}{z_{34}} \right) M_{1234} + \mathcal{P}(2,3,4)$$
$$= \prod_{k=2}^{4} \left(\sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right) M_{1234} + \mathcal{P}(2,3,4). \tag{5.53}$$

This gives us the appropriate generalization:

$$\frac{T_{123\ldots p}}{z_{123\ldots p}} + \mathcal{P}(1,2,3,\ldots,p) = \prod_{k=2}^{p} \left(\sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right) M_{123\ldots p} + \mathcal{P}(1,2,3,\ldots,p). \tag{5.54}$$

Terms of this form will be multiplying their counterparts in (5.45), which are of the form

$$\frac{T_{N-1,N-2,\ldots,p+1}}{z_{N-1,N-2,\ldots,p+1}} + \mathcal{P}(p+1,\ldots,N-2) = \prod_{k=p+1}^{N-2} \sum_{m=k+1}^{N-1} \frac{s_{km}}{z_{km}} M_{N-1,\ldots,p+1} + \mathcal{P}(p+1,\ldots,N-2). \tag{5.55}$$

The BG current in question is of rank-$N-p-2$, so reflecting the labels gives an overall sign of $(-1)^{N-p-3}$. The product contains $N-p-3$ factors of z_{km}, so reversing these to z_{mk} will cancel the sign. Then we can write

$$\frac{T_{N-1,N-2,\ldots,p+1}}{z_{N-1,N-2,\ldots,p+1}} + \mathcal{P}(p+1,\ldots,N-2) = \prod_{k=p+1}^{N-2} \sum_{m=k+1}^{N-1} \frac{s_{mk}}{z_{mk}} M_{p+1,\ldots,N-1} + \mathcal{P}(p+1,\ldots,N-2). \tag{5.56}$$

In the amplitude, we have terms of the form

$$\frac{T_{12\ldots p} T_{N-1,N-2,\ldots,p+1}}{z_{12\ldots p}\, z_{N-1,N-2,\ldots,p+1}} + \mathcal{P}(2,\ldots,N-2). \tag{5.57}$$

Each $T \to M$ replacement involves only the permutations of that building block's own labels, and we are summing over all permutations of all the labels $2\ldots N-2$. Thus for each set of labels $1\ldots p$, we sum over all permutations of $p+1\ldots N-2$, and *vice versa* (and also the same for all the other permutations). So despite our terms containing a product of two building blocks, the overall permutation sum guarantees that we can make the double replacement $TT \to MM$.

How do the kinematic factors resulting from the replacement behave when combined? Consider a kinematic integral like the one appearing in (5.54), with $p = N-2$:

$$\prod_{\ell=2}^{N-2} \int \mathrm{d}z_\ell \prod_{i<j} |z_{ij}|^{-s_{ij}} \prod_{k=2}^{N-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}}$$
$$= \int \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \left(\frac{s_{14}}{z_{14}} + \frac{s_{24}}{z_{24}} + \frac{s_{34}}{z_{34}} \right) \cdots \left(\frac{s_{1,N-2}}{z_{1,N-2}} + \frac{s_{2,N-2}}{z_{2,N-2}} + \cdots + \frac{s_{N-3,N-2}}{z_{N-3,N-2}} \right). \tag{5.58}$$

All the denominators z_{ij} are unique, meaning we are free to use the simple integration by parts to change the contents of any of the factors:

$$\sum_{m<k} \frac{s_{mk}}{z_{mk}} \to -\sum_{m>k} \frac{s_{mk}}{z_{mk}} = \sum_{m>k} \frac{s_{km}}{z_{km}}. \tag{5.59}$$

Suppose we make this exchange for every factor where $k > p$ for some p. Then (5.58) becomes

$$\begin{aligned}
&\int \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \cdots \left(\frac{s_{1p}}{z_{1p}} + \cdots + \frac{s_{p-1,p}}{z_{p-1,p}} \right) \left(\frac{s_{p+1,p+2}}{z_{p+1,p+2}} + \frac{s_{p+1,p+3}}{z_{p+1,p+3}} + \cdots + \frac{s_{p+1,N-1}}{z_{p+1,N-1}} \right) \cdots \\
&\qquad \times \cdots \left(\frac{s_{N-3,N-2}}{z_{N-3,N-2}} + \frac{s_{N-3,N-1}}{z_{N-3,N-1}} \right) \frac{s_{N-2,N-1}}{z_{N-2,N-1}} \\
&= \int \prod_{k=2}^{p} \left(\sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right) \prod_{\ell=p+1}^{N-2} \left(\sum_{n=\ell+1}^{N-1} \frac{s_{\ell n}}{z_{\ell n}} \right).
\end{aligned} \tag{5.60}$$

These two factors are exactly the kinematic factors accompanying the BG currents $M_{123\ldots p}$ and $M_{p+1,\ldots,N-1}$ after replacing the building blocks! So we find that even in the general case (5.45), after replacing $TT \to MM$, all the terms (at various p) within a given permutation end up with a common kinematic factor after integrating by parts. Using this result to transform the N-point amplitude (5.45), we finally find:

$$\begin{aligned}
\mathcal{A}_N &= \int \sum_{p=1}^{N-2} \left\langle \frac{T_{12\ldots p} T_{N-1,N-2,\ldots,p+1}}{z_{12\ldots p} z_{N-1,N-2,\ldots,p+1}} \right\rangle_N + \mathcal{P}(2,3,\ldots,N-2) \\
&= \int \sum_{p=1}^{N-2} \left\langle \prod_{k=2}^{p} \left(\sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right) \prod_{\ell=p+1}^{N-2} \left(\sum_{n=\ell+1}^{N-1} \frac{s_{\ell n}}{z_{\ell n}} \right) M_{12\ldots p} M_{p+1,\ldots,N-1} \right\rangle_N + \mathcal{P}(2,\ldots,N-2) \\
&= \left\langle \sum_{p=1}^{N-2} M_{12\ldots p} M_{p+1,\ldots,N-1} V^N \right\rangle \int \prod_{k=2}^{N-2} \left(\sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right) + \mathcal{P}(2,\ldots,N-2) \\
\mathcal{A}_N &= \mathcal{A}_{\mathrm{YM}}(1,2,\ldots N) \prod_{\ell=2}^{N-2} \int_{z_\ell < z_{\ell+1}} \mathrm{d}z_\ell \prod_{i<j} |z_{ij}|^{-s_{ij}} \prod_{k=2}^{N-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} + \mathcal{P}(2,\ldots,N-2).
\end{aligned} \tag{5.61}$$

5.5 Concluding Remarks

In this thesis we have reviewed the computation of the full N-point tree-level colour ordered open superstring amplitude using the Pure Spinor formalism and the calculational methods developed throughout [1, 3–7]. We focused on the technical details of the calculation, forgoing the kind of detailed analysis performed in [7]. In that paper, Mafra, Schlotterer, and Stieberger use integral techniques similar to those in section 5.2 to establish a basis for the kinematic integrals appearing in the N-point amplitude, drawing from a range of areas of mathematics. They find that such a basis is a valuable tool for studying a duality between colour and kinematics, which is known from field theory. Such a duality is made possible by various relations between subamplitudes which reduce the total number of independent subamplitudes down to the same number as the size of the basis of kinematic factors, as was discovered in [12].

Our focus on the tree-level also excluded the more recent one-loop analysis from our scope; this work has been performed in [20]. Similarly, our purely massless considerations could in principle be extended to include the first level, the vertex operators for which were introduced in [23]. Since these vertex operators contain many more terms than their massless counterparts considered in this thesis, some sort of computational algebra handling would be of great benefit in performing this calculation.

A Some Explicit OPEs

Some of the OPEs in this work were carried out using a Python script due to the large number of terms involved. The output of some of these calculations is reproduced here for reference purposes. This output was then simplified by hand using the SYM field equations and other identities, in particular the gamma matrix identity (mentioned in [5])

$$\mathrm{tr}(\gamma^{mn}\gamma_{pq}) = -32\delta^{mn}_{pq} \tag{A.1}$$

or with the help of the `GAMMA` package for Mathematica, published in [24].

For two integrated vertex operators, the script gives us

$$\begin{aligned}
&\lim_{z_3\to z_2} U^2(z_2)U^3(z_3) = \\
&\lim_{z_3\to z_2} (\partial\theta^\alpha A^2_\alpha + A^2_m\Pi^m + d_\alpha W^{2\,\alpha} + \frac{1}{2}N_{mn}\mathcal{F}^{2\,mn})(z_2)\cdots \\
&\qquad\qquad \times (\partial\theta^\beta A^3_\beta + A^3_p\Pi^p + d_\beta W^{3\,\beta} + \frac{1}{2}N_{pq}\mathcal{F}^{3\,pq})(z_3) \\
&\quad\to -\frac{\partial\theta^\alpha A^3_m k^{2\,m} A^2_\alpha}{(z_3-z_2)} + \frac{A^2_\alpha W^{3\,\beta}\delta^\beta_\alpha}{(z_3-z_2)^2} - \frac{\partial\theta^\alpha W^{3\,\beta} D_\beta A^2_\alpha}{(z_3-z_2)} \\
&\qquad + \frac{k^{3\,m} A^3_\beta A^2_m \partial\theta^\beta}{(z_2-z_3)} - \frac{\eta^{mn} A^2_m A^3_n}{(z_2-z_3)^2} - \frac{k^{3\,m} A^3_n A^2_m \Pi^n}{(z_2-z_3)} \\
&\qquad - \frac{\Pi^m A^3_n k^{2\,n} A^2_m}{(z_3-z_2)} + \frac{k^{3\,m} A^3_n k^{2\,n} A^2_m}{(z_2-z_3)(z_3-z_2)} + \frac{k^{3\,m} W^{3\,\beta} A^2_m d_\beta}{(z_2-z_3)} \\
&\qquad - \frac{A^2_m W^{3\,\beta}\gamma^m_{\beta\gamma}\partial\theta^\gamma}{(z_3-z_2)} - \frac{\Pi^m W^{3\,\beta} D_\beta A^2_m}{(z_3-z_2)} + \frac{k^{3\,m} W^{3\,\beta} D_\beta A^2_m}{(z_2-z_3)(z_3-z_2)} \\
&\qquad - \frac{1}{2}\frac{k^{3\,m}\mathcal{F}^3_{pn} A^2_m N^{pn}}{(z_2-z_3)} - \frac{\delta^\alpha_\beta W^{2\,\alpha} A^3_\beta}{(z_2-z_3)^2} + \frac{D_\alpha A^3_\beta W^{2\,\alpha}\partial\theta^\beta}{(z_2-z_3)} \\
&\qquad + \frac{\gamma^n_{\alpha\beta}\partial\theta^\beta W^{2\,\alpha} A^3_n}{(z_2-z_3)} + \frac{D_\alpha A^3_n W^{2\,\alpha}\Pi^n}{(z_2-z_3)} - \frac{d_\alpha A^3_n k^{2\,n} W^{2\,\alpha}}{(z_3-z_2)} \\
&\qquad - \frac{D_\alpha A^3_n k^{2\,n} W^{2\,\alpha}}{(z_2-z_3)(z_3-z_2)} - \frac{\gamma_{m\alpha\beta}\Pi^m W^{2\,\alpha} W^{3\,\beta}}{(z_2-z_3)} + \frac{D_\alpha W^{3\,\beta} W^{2\,\alpha} d_\beta}{(z_2-z_3)} \\
&\qquad - \frac{d_\alpha W^{3\,\beta} D_\beta W^{2\,\alpha}}{(z_3-z_2)} - \frac{D_\alpha W^{3\,\beta} D_\beta W^{2\,\alpha}}{(z_2-z_3)(z_3-z_2)} + \frac{1}{2}\frac{D_\alpha\mathcal{F}^3_{pn} W^{2\,\alpha} N^{pn}}{(z_2-z_3)}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\frac{N^{mn}A^3_p k^{2\,p}\mathcal{F}^2_{mn}}{(z_3-z_2)} - \frac{1}{2}\frac{N^{mn}W^{3\,\beta}D_\beta\mathcal{F}^2_{mn}}{(z_3-z_2)} + \frac{1}{2}\frac{1}{2}\frac{\mathcal{F}^2_{mn}N^{mp}\eta^{nq}\mathcal{F}^3_{pq}}{(z_2-z_3)} \\
&-\frac{1}{2}\frac{1}{2}\frac{\mathcal{F}^2_{mn}N^{mq}\eta^{np}\mathcal{F}^3_{pq}}{(z_2-z_3)} + \frac{1}{2}\frac{1}{2}\frac{\mathcal{F}^2_{mn}N^{nq}\eta^{mp}\mathcal{F}^3_{pq}}{(z_2-z_3)} - \frac{1}{2}\frac{1}{2}\frac{\mathcal{F}^2_{mn}N^{np}\eta^{mq}\mathcal{F}^3_{pq}}{(z_2-z_3)} \\
&-\frac{1}{2}\frac{1}{2}3\frac{\mathcal{F}^2_{mn}\eta^{np}\eta^{mq}\mathcal{F}^3_{pq}}{(z_2-z_3)^2} + \frac{1}{2}\frac{1}{2}3\frac{\mathcal{F}^2_{mn}\eta^{nq}\eta^{mp}\mathcal{F}^3_{pq}}{(z_2-z_3)^2}.
\end{aligned}
\tag{A.2}
$$

By hand, we can simplify this output to

$$
\begin{aligned}
U^2(z_2)U^3(z_3) \to\ & \frac{1}{z_{32}}\big[(W^2\gamma_m W^3)(\Pi^m + k^{23}_n N^{nm}) - N^{mn}\mathcal{F}^2{}_m{}^p\mathcal{F}^3_{np} \\
&\qquad +[A^3_m(W^2\gamma^m\partial\theta) + (k^3\cdot\Pi)(A^2W^3) + (A^2\cdot k^3)U^3 - (2\leftrightarrow 3)]\big] \\
&+ \frac{1+s_{23}}{z^2_{32}}\big[(A^2W^3) + (A^3W^2) - (A^2\cdot A^3)\big] \\
\equiv\ & \frac{1}{z_{32}}Y^{32} + \frac{1+s_{32}}{z^2_{32}}P^{32}
\end{aligned}
\tag{A.3}
$$

Similarly, we can compute e.g.

$$
\begin{aligned}
&V^1(z_1)\ (U^2U^3)(z_3) \\
&\to \frac{\lambda^\gamma k^{3\,m}A^3_n A^2_m k^{1\,n}A^1_\gamma}{(z_2-z_3)(z_3-z_1)} + \frac{\lambda^\gamma A^3_n k^{1\,m}A^1_\gamma k^{2\,n}A^2_m}{(z_3-z_2)(z_3-z_1)} \\
&\quad + \frac{\lambda^\gamma k^{3\,m}W^{3\,\beta}A^2_m D_\beta A^1_\gamma}{(z_2-z_3)(z_3-z_1)} - \frac{\lambda^\gamma W^{3\,\beta}k^{1\,m}A^1_\gamma D_\beta A^2_m}{(z_3-z_2)(z_3-z_1)} \\
&\quad + \frac{1}{2}\frac{1}{2}\frac{A^1_\gamma k^{3\,m}\mathcal{F}^3_{pn}A^2_m\gamma^{pn\gamma}_\alpha\lambda^\alpha}{(z_2-z_3)(z_3-z_1)} - \frac{\lambda^\gamma D_\alpha A^3_n W^{2\,\alpha}k^{1\,n}A^1_\gamma}{(z_2-z_3)(z_3-z_1)} \\
&\quad + \frac{\lambda^\gamma A^3_n D_\alpha A^1_\gamma k^{2\,n}W^{2\,\alpha}}{(z_3-z_2)(z_3-z_1)} - \frac{\lambda^\gamma\gamma_{m\alpha\beta}W^{2\,\alpha}k^{1\,m}A^1_\gamma W^{3\,\beta}}{(z_2-z_3)(z_3-z_1)} \\
&\quad + \frac{\lambda^\gamma D_\alpha W^{3\,\beta}W^{2\,\alpha}D_\beta A^1_\gamma}{(z_2-z_3)(z_3-z_1)} + \frac{\lambda^\gamma W^{3\,\beta}D_\alpha A^1_\gamma D_\beta W^{2\,\alpha}}{(z_3-z_2)(z_3-z_1)} \\
&\quad - \frac{1}{2}\frac{1}{2}\frac{A^1_\gamma D_\alpha\mathcal{F}^3_{pn}W^{2\,\alpha}\gamma^{pn\gamma}_\beta\lambda^\beta}{(z_2-z_3)(z_3-z_1)} + \frac{1}{2}\frac{1}{2}\frac{A^1_\gamma A^3_p\gamma^{mn\gamma}_\alpha\lambda^\alpha k^{2\,p}\mathcal{F}^2_{mn}}{(z_3-z_2)(z_3-z_1)} \\
&\quad + \frac{1}{2}\frac{1}{2}\frac{A^1_\gamma W^{3\,\beta}\gamma^{mn\gamma}_\alpha\lambda^\alpha D_\beta\mathcal{F}^2_{mn}}{(z_3-z_2)(z_3-z_1)} - \frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{A^1_\gamma\mathcal{F}^2_{mn}\eta^{nq}\gamma^{mp\gamma}_\alpha\lambda^\alpha\mathcal{F}^3_{pq}}{(z_2-z_3)(z_3-z_1)} \\
&\quad + \frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{A^1_\gamma\mathcal{F}^2_{mn}\eta^{np}\gamma^{mq\gamma}_\alpha\lambda^\alpha\mathcal{F}^3_{pq}}{(z_2-z_3)(z_3-z_1)} - \frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{A^1_\gamma\mathcal{F}^2_{mn}\eta^{mp}\gamma^{nq\gamma}_\alpha\lambda^\alpha\mathcal{F}^3_{pq}}{(z_2-z_3)(z_3-z_1)} \\
&\quad + \frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{A^1_\gamma\mathcal{F}^2_{mn}\eta^{mq}\gamma^{np\gamma}_\alpha\lambda^\alpha\mathcal{F}^3_{pq}}{(z_2-z_3)(z_3-z_1)} \\
&= \frac{L_{2331}}{z_{23}z_{31}}
\end{aligned}
\tag{A.4}
$$

using the output of the U^2U^3 calculation directly, or

$$
\begin{aligned}
&V^1(z_1)\ (U^2U^3)(z_3) \\
&\quad\to\ -\frac{\lambda^\alpha W^{2\,\gamma}\gamma_{m\gamma\beta}W^{3\,\beta}k^{1\,m}A^1_\alpha}{(z_2-z_3)(z_3-z_1)}-\frac{1}{2}\frac{A^1_\alpha W^{2\,\gamma}\gamma_{m\gamma\beta}W^{3\,\beta}k^2_n\gamma^{nm\alpha}_\delta\lambda^\delta}{(z_2-z_3)(z_3-z_1)} \\
&\qquad-\frac{1}{2}\frac{A^1_\alpha W^{2\,\gamma}\gamma_{m\gamma\beta}W^{3\,\beta}k^3_n\gamma^{nm\alpha}_\delta\lambda^\delta}{(z_2-z_3)(z_3-z_1)}+\frac{1}{2}\frac{A^1_\alpha\mathcal{F}^2_{mp}\gamma^{mn\alpha}_\beta\lambda^\beta\mathcal{F}^3_{np}}{(z_2-z_3)(z_3-z_1)} \\
&\qquad+\frac{\lambda^\delta W^{2\,\alpha}D_\alpha A^3_m k^{1\,m}A^1_\delta}{(z_2-z_3)(z_3-z_1)}+\frac{\lambda^\delta W^{2\,\alpha}D_\alpha W^{3\,\beta}D_\beta A^1_\delta}{(z_2-z_3)(z_3-z_1)} \\
&\qquad+\frac{\lambda^\delta A^2_m k^{3\,m}A^3_n k^{1\,n}A^1_\delta}{(z_2-z_3)(z_3-z_1)}+\frac{\lambda^\delta A^2_m k^{3\,m}W^{3\,\alpha}D_\alpha A^1_\delta}{(z_2-z_3)(z_3-z_1)} \\
&\qquad+\frac{1}{2}\frac{1}{2}\frac{A^1_\delta A^2_m k^{3\,m}\mathcal{F}^3_{pn}\gamma^{pn\delta}_\alpha\lambda^\alpha}{(z_2-z_3)(z_3-z_1)}+\frac{\lambda^\delta W^{3\,\alpha}D_\alpha A^3_m k^{1\,m}A^1_\delta}{(z_2-z_3)(z_3-z_1)} \\
&\qquad+\frac{\lambda^\delta W^{3\,\alpha}D_\alpha W^{3\,\beta}D_\beta A^1_\delta}{(z_2-z_3)(z_3-z_1)}+\frac{\lambda^\delta A^3_m k^{2\,m}A^2_n k^{1\,n}A^1_\delta}{(z_2-z_3)(z_3-z_1)} \\
&\qquad+\frac{\lambda^\delta A^3_m k^{2\,m}W^{2\,\alpha}D_\alpha A^1_\delta}{(z_2-z_3)(z_3-z_1)}+\frac{1}{2}\frac{1}{2}\frac{A^1_\delta A^3_m k^{2\,m}\mathcal{F}^2_{pn}\gamma^{pn\delta}_\alpha\lambda^\alpha}{(z_2-z_3)(z_3-z_1)} \\
&\quad=\frac{L_{2331}}{z_{23}z_{31}}
\end{aligned}
\tag{A.5}
$$

using the simplified expression (A.3) for U^2U^3.

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